Weighted Residual Methods
Formulation of FEM Model

• Several approaches can be used to transform the physical formulation of a problem to its finite element discrete analogue.

• If the physical formulation of the problem is described as a differential equation, then the most popular solution method is the **Method of Weighted Residuals**.

• If the physical problem can be formulated as the minimization of a functional, then the **Variational Formulation** is usually used.
Finite element method is used to solve physical problems
  Solid Mechanics
  Fluid Mechanics
  Heat Transfer
  Electrostatics
  Electromagnetism
  ....

Physical problems are governed by **differential equations** which satisfy
**Boundary conditions**
**Initial conditions**

One variable: Ordinary differential equation (ODE)
Multiple independent variables: Partial differential equation (PDE)
Physical problems

Axially loaded elastic bar

\[ A(x) = \text{cross section at } x \]

\[ b(x) = \text{body force distribution} \]

(\text{force per unit length})

\[ E(x) = \text{Young’s modulus} \]

\[ u(x) = \text{displacement of the bar at } x \]

Differential equation governing the response of the bar

\[
\frac{d}{dx} \left( AE \frac{du}{dx} \right) + b = 0; \quad 0 < x < L
\]

Second order differential equations

Requires 2 boundary conditions for solution
**Physical problems**

Axially loaded elastic bar

![Diagram of an axially loaded elastic bar with boundary conditions](image)

**Boundary conditions (examples)**

- \( u = 0 \) at \( x = 0 \)  Dirichlet/ displacement bc
- \( u = 1 \) at \( x = L \)
- \( u = 0 \) at \( x = 0 \)

\[
EA \frac{du}{dx} = F \quad \text{at} \quad x = L
\]

Neumann/ force bc

**Differential equation + Boundary conditions = Strong form of the “boundary value problem”**
Flexible string

\[ S = \text{tensile force in string} \]
\[ p(x) = \text{lateral force distribution} \]
\[ (\text{force per unit length}) \]
\[ w(x) = \text{lateral deflection of the string in the y-direction} \]

Differential equation governing the response of the bar

\[ S \frac{d^2 u}{dx^2} + p = 0; \quad 0 < x < L \]

Second order differential equations
Requires 2 boundary conditions for solution
Heat conduction in a fin

\[ A(x) = \text{cross section at } x \]
\[ Q(x) = \text{heat input per unit length per unit time [J/sm]} \]
\[ k(x) = \text{thermal conductivity [J/°C ms]} \]
\[ T(x) = \text{temperature of the fin at } x \]

Differential equation governing the response of the fin

\[ \frac{d}{dx} \left( Ak \frac{dT}{dx} \right) + Q = 0; \quad 0 < x < L \]

Second order differential equations
Requires 2 boundary conditions for solution
Physical problems

Heat conduction in a fin

Boundary conditions (examples)

\[ T = 0 \at x = 0 \]

\[ -k \frac{dT}{dx} = h \at x = L \]

Dirichlet/ displacement bc

Neumann/ force bc
Fluid flow through a porous medium (e.g., flow of water through a dam)

Differential equation

\[ \frac{d}{dx} \left( k \frac{d\varphi}{dx} \right) + Q = 0; \quad 0 < x < L \]

Physical problems

Boundary conditions (examples)

\[ \varphi = 0 \quad \text{at} \quad x = 0 \quad \text{Known head} \]

\[ -k \frac{d\varphi}{dx} = h \quad \text{at} \quad x = L \quad \text{Known velocity} \]

Second order differential equations
Requires 2 boundary conditions for solution
Physical problems

<table>
<thead>
<tr>
<th>Differential equation</th>
<th>Physical problem</th>
<th>Quantities</th>
<th>Constitutive law</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} \left( Ak \frac{dT}{dx} \right) + Q = 0 )</td>
<td>One-dimensional heat flow</td>
<td>( T = ) temperature, ( A = ) area, ( k = ) thermal conductivity</td>
<td>Fourier ( q = -k \frac{dT}{dx} )</td>
</tr>
<tr>
<td>( \frac{d}{dx} \left( AE \frac{du}{dx} \right) + b = 0 )</td>
<td>Axially loaded elastic bar</td>
<td>( u = ) displacement, ( A = ) area, ( E = ) Young’s modulus</td>
<td>Hooke ( \sigma = E \frac{du}{dx} )</td>
</tr>
<tr>
<td>( S \frac{d^2 w}{dx^2} + p = 0 )</td>
<td>Transversely loaded flexible string</td>
<td>( w = ) deflection, ( S = ) string force, ( p = ) lateral loading</td>
<td></td>
</tr>
<tr>
<td>( \frac{d}{dx} \left( AD \frac{dc}{dx} \right) + Q = 0 )</td>
<td>One-dimensional diffusion</td>
<td>( c = ) iron concentration, ( A = ) area, ( D = ) diffusion coefficient, ( Q = ) ion supply</td>
<td>Fick ( q = -D \frac{dC}{dx} )</td>
</tr>
</tbody>
</table>
**Physical problems**

### Table 4.1 Examples of second-order differential equations

<table>
<thead>
<tr>
<th>Differential equation</th>
<th>Physical problem</th>
<th>Quantities</th>
<th>Constitutive law</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{d}{dx} \left( A \gamma \frac{dV}{dx} \right) + Q = 0 ]</td>
<td>One-dimensional electric current</td>
<td>( V = \text{voltage} )</td>
<td>Ohm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( A = \text{area} )</td>
<td>( q = -\gamma \frac{dV}{dx} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \gamma = \text{electric conductivity} )</td>
<td>( q = \text{electric charge flux} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Q = \text{electric charge supply} )</td>
<td></td>
</tr>
<tr>
<td>[ \frac{d}{dx} \left( A \frac{D^2}{32\mu} \frac{dp}{dx} \right) + Q = 0 ]</td>
<td>Laminar flow in pipe (Poiseuille flow)</td>
<td>( p = \text{pressure} )</td>
<td>( q = -\left( \frac{D^2}{32\mu} \right) \frac{dp}{dx} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( A = \text{area} )</td>
<td>( q = \text{volume flux} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( D = \text{diameter} )</td>
<td>( q = \text{mean velocity} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \mu = \text{viscosity} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( Q = \text{fluid supply} )</td>
<td></td>
</tr>
</tbody>
</table>
Formulation of FEM Model

Observe:
1. All the cases we considered lead to very similar differential equations and boundary conditions.
2. In 1D it is easy to analytically solve these equations.
3. Not so in 2 and 3D especially when the geometry of the domain is complex: need to solve **approximately**.
4. We’ll learn how to solve these equations in 1D. The approximation techniques easily translate to 2 and 3D, no matter how complex the geometry.
Finite Element Method
Integral Formulation
Some Mathematical Concepts

Simply connected domain: If any two points of the domain can be joint by a line lying entirely within the domain.

Class of a domain: A function of several variables is said to be of Class $C^m(\Omega)$ in a domain if all its partial derivatives up to and including the $m$th order exist and are continuous in $\Omega$.

$C^0 \rightarrow$ $F$ is continuous (i.e. $\partial f / \partial x$, $\partial f / \partial y$ exist but may not be continuous.)

Boundary Value Problems: A differential equation (DE) is said to be a BVP if the dependent variable and possibly its derivatives are required to take specified values on the boundary.

Example: $\frac{d}{dx} \left( a \frac{du}{dx} \right) = f \quad 0 < x < 1, \quad u(0) = d_0, \left( x \frac{du}{dx} \right)_{x=1} = g_0$
Some Mathematical Concepts

**Initial Value Problem:** An IVP is one in which the dependent variable and possibly its derivatives are specified initially at \( t = 0 \)

Example:

\[
\rho \frac{d^2 u}{dt^2} + au = f \quad 0 < t \leq t_0, \quad u(0) = u_0, \left( \frac{du}{dt} \right)_{t=0} = v_0
\]

**Initial and Boundary Value Problem:**

Example:

\[
- \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) + \rho \frac{\partial u}{\partial t} = f(x,t) \quad \text{for} \quad 0 < x < 1 \text{and} \quad 0 < t \leq t_0
\]

\[
u(0,t) = d_0(t), \left( a \frac{\partial u}{\partial x} \right)_{x=1} = g_0(t), \quad u(x,0) = u_0(x)
\]

**Eigenvalue Problem:** the problem of determining value \( \lambda \) of such that

Example:

\[
\begin{align*}
\lambda & \quad \text{Eigenvalue} \\
u & \quad \text{Eigenfunction}
\end{align*}
\]

\[
- \frac{d}{dx} \left( a \frac{du}{dx} \right) = \lambda u \quad 0 < x < 1
\]

\[
u(0) = 0, \left( \frac{du}{dx} \right)_{x=1} = 0
\]
Some Mathematical Concepts

Integration-by-Part Formula:

First

\[
\frac{d}{dx}(wv) = \frac{dw}{dx}v + w\frac{dv}{dx} \Rightarrow \int_a^b w\frac{dv}{dx} \, dx = -\int_a^b v\frac{dw}{dx} \, dx + w(b)v(b) - w(a)v(a)
\]

Next

\[
\int_a^b w\frac{d^2u}{dx^2} \, dx = -\int_a^b \frac{du}{dx}\frac{dw}{dx} \, dx + w(b)\frac{du}{dx}(b) - w(a)\frac{du}{dx}(a)
\]

Similarly

\[
\int_a^b v\frac{d^4w}{dx^4} \, dx = \int_a^b \frac{d^2w}{dx^2}\frac{d^2v}{dx^2} \, dx + \frac{d^2w}{dx^2}(a)\frac{dv}{dx}(a) - \frac{d^2w}{dx^2}(b)\frac{dv}{dx}(b) + v(b)\frac{d^3w}{dx^3}(b) - v(a)\frac{d^3w}{dx^3}(a)
\]
Some Mathematical Concepts

**Gradient Theorem**

\[ \int_{\Omega} \text{grad } F \, dx \, dy = \int_{\Omega} \nabla F \, dx \, dy = \oint_{\Gamma} \hat{n} F \, ds \]

But

\[ \nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j, \quad \hat{n} = n_x i + n_y j \]

Thus

\[ \int_{\Omega} \left( \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j \right) dx \, dy = \oint_{\Gamma} \left( n_x i + n_y j \right) F \, ds \]

or

\[ \int_{\Omega} \left( \frac{\partial F}{\partial x} \right) dx \, dy = \oint_{\Gamma} F n_x \, ds \]

\[ \int_{\Omega} \left( \frac{\partial F}{\partial y} \right) dx \, dy = \oint_{\Gamma} F n_y \, ds \]
Some Mathematical Concepts

**Divergence Theorem**

\[
\int_{\Omega} \text{div} G \, dx\,dy = \int_{\Omega} \nabla \cdot G \, dx\,dy = \oint_{\Gamma} \hat{n} \cdot G \, ds
\]

\[
\int_{\Omega} \left( \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} \right) dx\,dy = \oint_{\Gamma} (n_x G_x + n_y G_y) \, ds
\]

Using gradient and divergence theorem, the following relations can be derived! (Exercise)

\[
\int_{\Omega} (\nabla G) w \, dx\,dy = -\int_{\Omega} (\nabla w) G \, dx\,dy + \oint_{\Gamma} \hat{n} w G \, ds
\]

\[
\int_{\Omega} (\nabla^2 G) w \, dx\,dy = \int_{\Omega} (\nabla w) \cdot (\nabla G) \, dx\,dy - \oint_{\Gamma} \frac{\partial G}{\partial n} \, w ds
\]

\((*)\) and
Some Mathematical Concepts

The components of equation (*) are:

\[
\int_{\Omega} \frac{\partial G}{\partial x} w \, dx \, dy = - \int_{\Omega} \frac{\partial w}{\partial x} G \, dx \, dy + \oint_{\Gamma} n_x w G \, ds
\]

\[
\int_{\Omega} \frac{\partial G}{\partial y} w \, dx \, dy = - \int_{\Omega} \frac{\partial w}{\partial y} G \, dx \, dy + \oint_{\Gamma} n_y w G \, ds
\]
Some Mathematical Concepts

**Functionals**

An integral in the form of

$$I(u) = \int_a^b F(x,u,u')\,dx, \quad u = u(x), \quad u' = \frac{du}{dx}$$

where integrand $F(x,u,u')$ is a given function of arguments $x, u, u'$ is called a *functional* (a function of function).

A functional is said to be **linear** if and only if:

$$I(\alpha u + \beta v) = \alpha I(u) + \beta I(v) \quad \alpha, \beta \text{ are scalars}$$

A functional $B(u,v)$ is said to be **bilinear** if it is linear in each of its arguments

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) \quad \text{Linearity in the first argument}$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2) \quad \text{Linearity in the second argument}$$
Some Mathematical Concepts

Functionals

A *bilinear* form \( B(u,v) \) is *symmetric* in its arguments if

\[
B(u,v) = B(v,u)
\]

Example of linear functional is

\[
I(v) = \int_0^L v f dx + \frac{d v}{d x} (L) M_0
\]

Example of bilinear functional is

\[
B(v,w) = \int_0^L a \frac{d v}{d x} \frac{d w}{d x} dx
\]
### 4.4.1 The Variational Operator

The delta operator $\delta$ used in conjunction with virtual quantities has special importance in variational methods. The operator is called the *variational operator* because it is used to denote a variation (or change) in a given quantity. In this section, we discuss certain operational properties of $\delta$ and elements of variational calculus. Using these tools, we can study the energy and variational principles of general problems.

Let $u = u(x)$ be the true configuration (i.e., the one corresponding to equilibrium) of a given mechanical system, and suppose that $u = \hat{u}$ on boundary $S_1$ of the total boundary $S$. Then an admissible configuration is of the form

$$\bar{u} = u + \alpha v$$

(4.62)

everywhere in the body, where $v$ is an arbitrary function that satisfies the homogeneous geometric boundary condition of the system

$$v = 0 \quad \text{on } S_1.$$  

(4.63)
the space of admissible variations, as already mentioned. Figure 4.13 shows a typical competing function \( \bar{u}(x) = u(x) + \alpha v(x) \) and a typical admissible variation \( v(x) \).
Some Mathematical Concepts

Here $\alpha v$ is a variation of the given configuration $u$. It should be understood that the variations are small enough (i.e., $\alpha$ is small) not to disturb the equilibrium of the system, and the variation is consistent with the geometric constraint of the system. Equation (4.62) defines a set of varied configurations; an infinite number of configurations $\bar{u}$ can be generated for a fixed $v$ by assigning values to $\alpha$. All of these configurations satisfy the specified geometric boundary conditions on boundary $S_1$, and therefore they constitute the set of admissible configurations. For any $v$, all configurations reduce to the actual one when $\alpha$ is zero. Therefore for any fixed $x$, $\alpha v$ can be viewed as a change or variation in the actual configuration $u$. This variation is often denoted by $\delta u$:

$$
\delta u = \alpha v, \quad \delta\left(\frac{du}{dx}\right) = \alpha\left(\frac{dv}{dx}\right) = \frac{d(\alpha v)}{dx} = \frac{d\delta u}{dx}, \tag{4.64}
$$

and $\delta u$ is called the first variation of $u$. 

Next, consider a function of the dependent variable $u$ and its derivative $u' = \frac{du}{dx}$:

$$F = F(x, u, u').$$  \hspace{1cm} (4.65)

For fixed $x$, the change in $F$ associated with a variation in $u$ (and hence $u'$) is

$$\Delta F = F(x, u + \alpha v, u' + \alpha v') - F(x, u, u')$$

$$= F(x, u, u') + \frac{\partial F}{\partial u} \alpha v + \frac{\partial F}{\partial u'} \alpha v'$$

$$+ \frac{(\alpha v)^2}{2!} \frac{\partial^2 F}{\partial u^2} + \frac{2(\alpha v)(\alpha v')}{2!} \frac{\partial^2 F}{\partial u \partial u'} + \cdots - F(x, u, u')$$

$$= \frac{\partial F}{\partial u} \alpha v + \frac{\partial F}{\partial u'} \alpha v' + O(\alpha^2),$$  \hspace{1cm} (4.66)
where $O(\alpha^2)$ denotes terms of order $\alpha^2$ and higher. The first total variation of $F(x, u, u')$ is defined by

$$\delta F = \alpha \left[ \lim_{\alpha \to 0} \frac{\Delta F}{\alpha} \right]$$

$$= \alpha \left( \frac{\partial F}{\partial u} v + \frac{\partial F}{\partial u'} v' \right)$$

$$= \frac{\partial F}{\partial u} \alpha v + \frac{\partial F}{\partial u'} \alpha v'$$

$$= \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'.$$  \hspace{1cm} (4.67a)
Alternatively, the first variation may be defined as

$$
\delta F = \alpha \left[ \frac{dF(u + \alpha \nu, u' + \alpha \nu')}{d\alpha} \right]_{\alpha=0}
= \frac{\partial F}{\partial u} \alpha \nu + \frac{\partial F}{\partial u'} \alpha \nu' \\
= \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'.
$$

There is an analogy between the first variation of $F$ and the total differential of $F$. The total differential of $F$ is

$$
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial u} \, du + \frac{\partial F}{\partial u'} \, du'.
$$
Some Mathematical Concepts

If $G = G(u, v, w)$ is a function of several dependent variables (and possibly their derivatives), the total variation is the sum of partial variations:

$$\delta G = \delta_u G + \delta_v G + \delta_w G,$$

(4.70)

where, for example, $\delta_u$ denotes the partial variation with respect to $u$. The variational operator can be interchanged with differential and integral operators:

1. $$\delta \left( \frac{du}{dx} \right) = \alpha \frac{dv}{dx} = \frac{d}{dx} (\alpha v) = \frac{d}{dx} (\delta u).$$

2. $$\delta \left( \int_0^a u \, dx \right) = \alpha \int_0^a v \, dx = \int_0^a \alpha v \, dx = \int_0^a \delta u \, dx.$$
Some Mathematical Concepts

The Variational Symbol

Consider the function $F = F(x,u,u')$ for fixed value of $x$, $F$ only depends on $u,u'$

The change $\alpha \nu$ in $u$, where $\alpha$ is constant and $\nu$ is a function, is called variation of $u$ and denoted by:

$\delta u = \alpha \nu$

In analogy with the total differential of a function

$\delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$

Note that

$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial u'} du'$
Some Mathematical Concepts

The Variational Symbol

Also
\[ \delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2 \]
\[ \delta(F_1 F_2) = F_2 \delta F_1 + F_1 \delta F_2 \]
\[ \delta \left( \frac{F_1}{F_2} \right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2} \]
\[ \delta \left[ (F_1)^n \right] = n(F_1)^{n-1} \delta F_1 \]

Furthermore
\[ \frac{d}{dx} (\delta u) = \frac{d}{dx} (\alpha v) = \alpha \frac{dv}{dx} = \alpha v' = \delta u' = \delta \left( \frac{du}{dx} \right) \]
\[ \delta \int_a^b u(x) dx = \int_a^b \delta u(x) dx \]
Weak Formulation of BVP

Weighted – integral and weak formulation

Consider the following DE

\[- \frac{d}{dx} \left[ a(x) \frac{du}{dx} \right] = q(x) \quad 0 < x < L\]

\[u(0) = u_0, \quad \left( a \frac{du}{dx} \right)_{x=L} = Q_0\]

Transverse deflection of a cable
Axial deformation of a bar
Heat transfer
Flow through pipes
Flow through porous media
Electrostatics
There are 3 steps in the development of a weak form, if exists, of any DE.

**STEP 1:**
Move all expression in DE to one side, multiply by $w$ (weight function) and integrate over the domain.

$$\int_{0}^{L} w \left[ -\frac{d}{dx} \left( a \frac{du}{dx} \right) - q \right] \, dx = 0$$

Weighted-integral or weighted-residual

$$u = U_N = \sum_{j=1}^{N} c_j \phi_j + \phi_0$$

$N$ linearly independent equation for $w$ and obtain $N$ equation for $c_1, \ldots, c_N$
Weak Formulation of BVP

STEP 2

1-The integral (+) allows to obtain \( N \) independent equations
2- The approximation function, \( \phi \), should be differentiable as many times as called for the original DE.
3- The approximation function should satisfy the BCs.
4- If the differentiation is distributed between \( w \) and \( \phi \) then the resulting integral form has weaker continuity conditions. Such a weighted-integral statement is called weak form.

The weak form formulation has two main characteristics:
- requires weaker continuity on the dependent variable and often results in a symmetric set of algebraic equations.
- The natural BCs are included in the weak form, and therefore the approximation function is required to satisfy only the essential BCs.
Returning to our example:

\[
\int_0^L \left\{ w \left[ -\frac{d}{dx} \left( a \frac{du}{dx} \right) \right] - wq \right\} dx = 0 \Rightarrow \int_0^L \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[ wa \frac{du}{dx} \right]_0^L = 0
\]

**Secondary Variable (SV):** Coefficient of weight function and its derivatives

\[ Q = (a \frac{du}{dx})n_x \]  

**Natural Boundary Conditions (NBC):**

**Primary Variable (PV):** The dependent variable of the problem

\[ u \]  

**Essential Boundary Conditions (EBC):**
Weak Formulation of BVP

\[
\int_0^L \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[ wa \frac{du}{dx} \right]_0^L = 0
\]

\[
\int_0^L \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[ wa \frac{du}{dx} n_x \right]_{x=0}^L - \left[ wa \frac{du}{dx} n_x \right]_{x=L} = 0
\]

\[
\int_0^L \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - (wQ)_0 - (wQ)_L = 0
\]

Note that

\[
\begin{align*}
n_x &= -1 \quad x = 0 \\
n_x &= 1 \quad x = L
\end{align*}
\]
Weak Formulation of BVP

STEP 3:
The last step is to impose the actual BCs of the problem $w$ has to satisfy the \textit{homogeneous form} of specified EBC.

In weak formulation $w$ has the meaning of a virtual change in PV. If PV is specified at a point, its variation is zero.

$$u(0) = u_0 \Rightarrow w(0) = 0$$

Thus

$$\left( a \frac{du}{dx} n_x \right)_{x=L} = \left( a \frac{du}{dx} \right)_{x=L} = Q_0 \text{ NBC}$$

$$\int_{0}^{L} \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - \left[ wa \frac{du}{dx} n_x \right]_{x=0} - \left[ wa \frac{du}{dx} n_x \right]_{x=L} = 0$$

$$\int_{0}^{L} \left( \frac{dw}{dx} a \frac{du}{dx} - wq \right) dx - w(L)Q_0 = 0$$
Linear and Bilinear Forms

\[ \int_{0}^{L} \left( \frac{dw}{dx} a \frac{du}{dx} \right) dx - \int_{0}^{L} wq dx - w(L)Q_0 = 0 \]

\[ B(w, u) - l(w) = 0 \]

\( B(w, u) \) Bilinear and symmetric in \( w \) and \( u \)

\( l(w) \) Linear

Therefore, problem associated with the DE can be stated as one of finding the solution \( u \) such that \( B(w, u) = l(w) \)

holds for any \( w \) satisfies the homogeneous form of the EBC and continuity condition implied by the weak form.
Linear and Bilinear Forms

Assume

\[ u = u^* + w \]

Satisfy the homogeneous Form of EBC

Variational solution
Satisfy EBC

Actual solution
Satisfy EBC+NBC

Looking at the definition of the variational symbol, \( w \) is the variation of the solution, i.e.

\[ w = \delta u \]

Then

\[ B(w, u) = l(w) \Rightarrow B(\delta u, u) = l(\delta u) \]  (#)

\[
B(\delta u, u) = \int_0^L a \frac{d\delta u}{dx} \frac{du}{dx} \, dx = \delta \int_0^L a \left[ \frac{1}{2} \left( \frac{du}{dx} \right)^2 \right] \, dx = \frac{1}{2} \delta \int_0^L a \frac{du}{dx} \frac{du}{dx} \, dx = \frac{1}{2} \delta [B(u, u)]
\]

\[
l(\delta u) = \int_0^L \delta u q \, dx + \delta u(L)Q_0 = \delta \left[ \int_0^L u q \, dx + u(L)Q_0 \right] = \delta [l(u)]
\]
Substituting in (#), we have:

\[ B(\delta u, u) - l(\delta u) = 0 \implies \delta \left[ \frac{1}{2} B(u, u) - l(u) \right] = 0 \implies \delta I(u) = 0 \]

\[ I(u) = \frac{1}{2} B(u, u) - l(u) \]  

(##)

In general, the relation \( B(\delta u, u) = \frac{1}{2} \delta B(u, u) \) holds only if \( B(w, u) \) is bilinear and symmetric and \( l(w) \) is linear.

If \( B(w, u) \) is not linear but symmetric the functional \( I(u) \) can be derived but not from (##). (see Oden & Reddy, 1976, Reddy 1986)
Linear and Bilinear Forms

Equation \( \delta I(u) = 0 \) represents the necessary condition for the functional \( I(u) \) to have an extremum value. For solid mechanics, \( I(u) \) represents the total potential energy functional and the statement of the total potential energy principle.

Of all admissible function \( u \), that which makes the total potential energy \( I(u) \) a minimum also satisfies the differential equation and natural boundary condition in (+).
Some Examples

Example 1

Consider the following DE which arise in the study of the deflection of a cable or heat transfer in a fin (when \( c = 0 \)).

\[
- \frac{d}{dx} \left( a \frac{du}{dx} \right) - cu + x^2 = 0 \quad \text{for} \quad 0 < x < 1
\]

\[
u(0) = 0, \quad \left( a \frac{du}{dx} \right)_{x=1} = 1
\]

Step 1

\[
\int_{0}^{1} w \left[ - \frac{d}{dx} \left( a \frac{dw}{dx} \right) - cu + x^2 \right] dx = 0
\]

Step 2

\[
\int_{0}^{1} \left( a \frac{dw}{dx} \frac{du}{dx} - cuw + wx^2 \right) dx - \left( wa \frac{du}{dx} \right)_{0}^{1} = 0
\]

\[
u(0) = 0 \quad \text{EBC}
\]

\[
\left( a \frac{du}{dx} \right)_{x=1} = 1 \quad \text{NBC}
\]

\[
w(0) = 0
\]
Some Examples

- Example 1

Step 3

\[
\int_0^1 \left( a \frac{dw}{dx} \frac{du}{dx} - cuw \right) dx + \int_0^1 wx^2 dx - w(1) = 0
\]

or

\[
B(w, u) = \int_0^1 \left( a \frac{dw}{dx} \frac{du}{dx} - cuw \right) dx
\]

\[
l(w) = -\int_0^1 wx^2 dx + w(1)
\]

\(B\) is bilinear and symmetric and \(l\) is linear! (prove)

Thus we can compute the quadratic functional form

\[
I(u) = \frac{1}{2} \int_0^1 \left( a \left( \frac{du}{dx} \right)^2 - cu^2 + 2ux^2 \right) dx - u(1)
\]
Some Examples

Example 2

Consider the following fourth-order DE (elastic bending of beam)

\[
\frac{d^2}{dx^2} \left( b \frac{d^2 w}{dx^2} \right) - f(x) = 0 \quad \text{for} \quad 0 < x < L
\]

\[w(0) = \frac{dw(0)}{dx} = 0, \quad \left( b \frac{d^2 w}{dx^2} \right)_{x=L} = M_0, \quad \frac{d}{dx} \left( b \frac{d^2 w}{dx^2} \right)_{x=L} = 0\]

Step 1

\[\int_{0}^{L} \left[ \frac{d^2}{dx^2} \left( b \frac{d^2 w}{dx^2} \right) - f \right] dx = 0\]

Step 2

\[\int_{0}^{L} \left[ - \frac{dv}{dx} \frac{d}{dx} \left( b \frac{d^2 w}{dx^2} \right) - vf \right] dx + \left[ v \frac{d}{dx} \left( b \frac{d^2 w}{dx^2} \right) \right]_{0}^{L} = 0\]
Some Examples

Example 2

\[
\int_0^L \left( b \frac{d^2v}{dx^2} \frac{d^2w}{dx^2} - vf \right) dx + \left[ v \frac{d}{dx} \left( b \frac{d^2w}{dx^2} \right) - \frac{dv}{dx} b \frac{d^2w}{dx^2} \right]_0^L = 0
\]

\[
\frac{d}{dx} \left( b \frac{d^2w}{dx^2} \right) = V \quad (Shear \ force)
\]

\[
b \frac{d^2w}{dx^2} = M \quad (Bending \ moment)
\]

B.C

\[
w(0) = \frac{dw(0)}{dx} = 0
\]

\[
\nu(0) = \frac{dv(0)}{dx} = 0
\]

\[
\frac{d}{dx} \left( b \frac{d^2w}{dx^2} \right)_{x=L} = 0
\]

\[
\left( b \frac{d^2w}{dx^2} \right)_{x=L} = M_0
\]
Some Examples

Example 2

Step 3

\[
\int_0^L \left( b \frac{d^2v}{dx^2} \frac{d^2w}{dx^2} - vf \right) dx - \left[ \frac{dv}{dx} \right]_{x=L} = M_0 = 0
\]

\[
B(v, w) = \int_0^L \left( b \frac{d^2v}{dx^2} \frac{d^2w}{dx^2} \right) dx
\]

or

\[
B(v, w) = l(v)
\]

where

\[
l(v) = \int_0^L vfdx + \left[ \frac{dv}{dx} \right]_{x=L}
\]

Symmetric & Bilinear

Linear

The functional \( I(w) \) can be written as:

\[
I(w) = \int_0^L \left[ b \left( \frac{d^2w}{dx^2} \right)^2 - w f \right] dx + \left[ \frac{dw}{dx} \right]_{x=L} = M_0
\]
Some Examples

**Example 3** Steady heat conduction in a two-dimensional domain $\Omega$

Consider a 2D heat transfer problem

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = q_0 \quad \text{in} \quad \Omega$$

$q_0$: uniform heat generation

$k$: conductivity of the isotropic material

$T$: temperature

---

Convection

Steady heat conduction in a two-dimensional domain $\Omega$

Insulated

$$\hat{q} = 0$$

$$k \frac{\partial T}{\partial x} = -\beta(T - T_\infty)$$

$$T = T_0(x)$$

---

$\partial T \over \partial n = -\partial T \over \partial x$

$k \frac{\partial T}{\partial x} = \hat{q}(y)$
Some Examples

Example 3

Step 1

\[ \int_{\Omega} w \left[ -k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) - q_0 \right] dx dy = 0 \]

Step 2

\[ \int_{\Omega} \left[ k \left( \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial T}{\partial y} \right) - wq_0 \right] dx dy - \int_{\Gamma} w k \left( \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) ds = 0 \] (*)

\[ k \left( \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) = k \frac{\partial T}{\partial n} = q_n \]

T = Primary variable

\[ q_n = \text{Secondary variable (heat flux)} \]

on \( \Gamma_1 = AB \ (n_x = -1, n_y = 0) \Rightarrow \hat{q}(y) \)

on \( \Gamma_2 = BC \ (n_x = 0, n_y = -1) \Rightarrow T_0(x) \)

on \( \Gamma_3 = CD \ (n_x = 1, n_y = 0) \Rightarrow k \frac{\partial T}{\partial n} + \beta(T - T_\infty) = 0 \)

on \( \Gamma_4 = DA \ (n_x = 0, n_y = 1) \Rightarrow \frac{\partial T}{\partial n} = 0 \)
Some Examples

Example 3

Step 3

\[ \int_{\Gamma} wk \left( \frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) ds = \int_{\Gamma} wk \left( \frac{\partial T}{\partial n} \right) ds = \]

\[ \int_{\Gamma_1} wq_n ds + \int_{\Gamma_2} 0k \left( \frac{\partial T}{\partial n} \right) ds - \int_{\Gamma_3} w[\beta(T - T_\infty)] ds + \int_{\Gamma_4} w(0) ds = \]

\[ - \int_0^b w(0, y) \hat{q}(y) dy - \beta \int_0^b w(a, y) [T(a, y) - T_\infty] dy \]

**w** should satisfy the EBC

Substituting in (*) we have

\[ \int_{\Omega} \left[ k \left( \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial T}{\partial y} \right) - wq_0 \right] dxdy + \int_0^b w(0, y) \hat{q}(y) dy + \beta \int_0^b w(a, y) [T(a, y) - T_\infty] dy = 0 \]

\[ B(w, T) = l(w) \]
Some Examples

Example 3

\[ B(w, T) = \int_{\Omega} \left[ k \left( \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial T}{\partial y} \right) \right] dx \, dy + \beta \int_{0}^{b} w(a, y) T(a, y) \, dy \]

\[ l(w) = \int_{\Omega} w q_0 dx \, dy - \int_{0}^{b} w(0, y) \hat{q}(y) \, dy + \beta \int_{0}^{b} w(a, y) T_\infty \, dy \]

The quadratic functional is given by:

\[ I(T) = \frac{k}{2} \int_{\Omega} \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right] dx \, dy - \int_{\Omega} T q_0 dx \, dy + \int_{0}^{b} T(0, y) \hat{q}(y) \, dy + \beta \int_{0}^{b} \frac{1}{2} \left[ T^2(a, y) - 2T(a, y) T_\infty \right] \, dy \]
Conclusions

1- The weak form of a DE is the same as the statement of the total potential energy.
2- Outside solid mechanics $I(u)$ may not have meaning of energy but it is still a use mathematical tools.
3- Every DE admits a weighted-integral statement, or a weak form exists for every DE of order two or higher.
4- Not every DE admits a functional formulation. For a DE to have a functional formulation, its bilinear form should be symmetric in its argument.
5- Variational or FE methods do not require a functional, a weak form of the equation is sufficient.
6- If a DE has a functional, the weak form is obtained by taking its first variation.
References
