Two-Parameter Fracture Mechanics
J integral-controlled fracture

Small-scale yielding

**LEFM**: small-scale yielding satisfied and generally have

\[ \sigma_{eq} \ll \sigma_{YS} \]

Relevant parameters:
- \( G \) (energy)
- \( K \) (stress)

Elastic-plastic conditions

**EPFM**: small-scale yielding is gradually violated and

\[ \sigma_{eq} \approx \sigma_{YS} \]

Relevant parameters:
- \( J \) (energy & used for stress)

Large-scale yielding

**Large-scale yielding condition**: No single parameter can characterize fracture!
- \( J \) + other parameters (e.g. \( T \) stress, \( Q-J \), etc)
J-Controlled Crack Growth

The material directly in front of the crack violates the single-parameter assumption because the loading is highly nonproportional, i.e., the various stress components increase at different rates and some components actually decrease. In order for the crack growth to be *J controlled*, the elastic unloading and nonproportional plastic loading regions must be embedded within a zone of *J dominance*. When the crack grows out of the zone of *J dominance*, the measured *R* curve is no longer uniquely characterized by *J*.

In small-scale yielding, there is always a zone of *J dominance* because the crack-tip conditions are defined by the elastic stress intensity, which depends only on the current values of the load and crack size. The crack never grows out of the *J-dominated zone* as long as all the specimen boundaries are remote from the crack tip and the plastic zone.
Actually, the plastic strain concentrations depend on the experiment which might be of the forms depicted in following pictures. It appears that the plastic zones are not reproducible from one test to another. Regarding the crack initiation criterion, we can say that the solution is no longer uniquely governed by $J$. The relation between $J$ and $\delta_t$ is dependent on the configuration and on the loading. The critical $J_C$ measured for an experiment might not be valid for another one. A two-parameter characterization is thus required.
\section*{T-stress}

\section*{The Elastic T Stress}

- Williams showed that the crack tip stress fields in an isotropic elastic material can be expressed as an infinite power series:

\[
\sigma_{rr} = \frac{1}{\sqrt{2\pi r}} \left\{ K_I \left[ \frac{5\cos \theta}{2} - \cos \frac{3\theta}{2} \right] + K_{II} \left[ -5\sin \frac{\theta}{2} + 3\sin \frac{3\theta}{2} \right] \right\} + T \cos^2 \theta + O(r^{1/2}) + \ldots
\]

\[
\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \left\{ K_I \left[ 3\cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right] + K_{II} \left[ -3\sin \frac{\theta}{2} - 3\sin \frac{3\theta}{2} \right] \right\} + T \sin^2 \theta + O(r^{1/2}) + \ldots
\]

\[
\sigma_{r\theta} = \frac{1}{\sqrt{2\pi r}} \left\{ K_I \left[ \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right] + K_{II} \left[ \cos \frac{\theta}{2} + 3\cos \frac{3\theta}{2} \right] \right\} - T \sin \theta \cos \theta + O(r^{1/2}) + \ldots
\]

- Although the third and higher terms in the Williams solution, which have positive exponents on \( r \), vanish at the crack tip, the second (uniform) term remains finite. It turns out that this second term can have a profound effect on the plastic zone shape and the stresses deep inside the plastic zone.

\[
\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_i(\theta) + \begin{bmatrix}
T & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \nu T
\end{bmatrix}
\]

\( T \) is a uniform stress in the \( x \) direction, which induces a stress \( T \) in the \( z \) direction in plane strain.
A modified boundary layer analysis

- The first two terms of the Williams series are applied as boundary conditions:

- Stress fields obtained from modified boundary layer analysis:

  Plastic analysis: $\sigma_{yy}$ is redistributed!

  High negative T stress:
  - Decreases $\sigma_{yy}$
  - Decreases triaxiality

  Positive T stress:
  - Slightly Increases $\sigma_{yy}$ and increase triaxiality
Higher order terms in stress expansion:

- **T-stress** (linear analysis)
  
  * Constant $\sigma_{xx}$ in LEFM expansion.
  * T stress redistributes stress field
  * Nondimensional biaxiality ratio: $\beta = \frac{T\sqrt{\pi a}}{K_I}$
  
  * for example $\beta = -1$ for mode-I crack in infinite domain.
  * $\beta$ depend on particular geometry/loading configuration
  * Effect of $T(\beta)$ on toughness:
    
    High (+) $T$ $\rightarrow$ constrained (triaxial) stress $\rightarrow$ Toughness $\downarrow$ Ductility $\downarrow$
    
    Low (-) $T$ $\rightarrow$ Lose constraint $\rightarrow$ Toughness $\uparrow$ Ductility $\uparrow$
  
  * T stress also influences crack path stability (particularly in dynamic fracture)
**J–Q Theory**

- **Q parameter (J–Q Theory)** valid for *nonlinear* analysis
- Added as a *hydrostatic shift* in front of crack to (HRR) stress fields:

\[
\sigma_{ij} \approx (\sigma_{ij})_{T=0} + Q \sigma_0 \delta_{ij} \quad \left( |\theta| \leq \frac{\pi}{2} \right)
\]

- Similar to *T* positive *Q* increases triaxiality and reduces fracture resistance

\[
J_c = J_c (Q)
\]

High (+) *Q* ➔ constrained (triaxial) stress ➔ Toughness ➔ Ductility ➔
Low (-) *Q* ➔ Lose constraint ➔ Toughness ➔ Ductility ➔

- **More number of parameters**: with extensive deformation two-parameter models such as *K, T* or *J, Q* eventually break.
**J–Q Theory**

> **Q defined as:**
>
>
>
>
>
>
>
>
>

\[
Q \equiv \frac{\sigma_{yy} - (\sigma_{yy})_{T=0}}{\sigma_0} \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \frac{r\sigma_0}{J} = 2
\]

Referring to Figure *, we see that \( Q \) is negative when \( T \) is negative.

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**Relationship between \( Q \) and \( T \)**

**Evolution of the \( Q \) parameter with deformation in two geometries**
The J–Q Toughness Locus

- in J–Q theory, an additional degree of freedom has been introduced, which implies that the critical J value for a given material depends on Q:

\[ J_c = J_c(Q) \]

The fracture toughness is no longer viewed as a single value; rather, it is a curve that defines a critical locus of J and Q values.

J–Q toughness locus for SE(B) specimens of A515 Grade 70 steel.
J–Q Theory

- The J–Q Toughness Locus

Single-parameter fracture mechanics theory assumes that toughness values obtained from laboratory specimens can be transferred to structural applications. Two-parameter approaches such as J–Q theory imply that the laboratory specimen must match the constraint of the structure; that is, the two geometries must have the same $Q$ at failure in order for the respective $J_c$ values to be equal. The figure illustrates the application of the J–Q approach to structures. The applied $J$ versus $Q$ curve for the configuration of interest is obtained from finite element analysis and plotted with the J–Q toughness locus. Failure is predicted when the driving force curve passes through the toughness locus. Since toughness data are often scattered, however, there is not a single unambiguous cross-over point. Rather, there is a range of possible $J_c$ values for the structure.

Application of a J–Q toughness locus. Failure occurs when the applied J–Q curve passes through the toughness locus.
Limitations of Two-Parameter Fracture Mechanics

The $T$ stress approach, $J$–$Q$ theory are examples of two-parameter fracture theories, where a second quantity (e.g., $T$, $Q$) has been introduced to characterize the crack tip environment. These approaches assume that the crack tip fields contain two degrees of freedom. When single-parameter fracture mechanics is valid, the crack tip fields have only one degree of freedom. In such cases, any one of several parameters (e.g., $J$, $K$, or CTOD) will suffice to characterize the crack tip conditions, provided the parameter can be defined unambiguously; $K$ is a suitable characterizing only when an elastic singularity zone exists ahead of the crack tip. Similarly, the choice of a second parameter in the case of two-parameter theory is mostly arbitrary, but the $T$ stress has no physical meaning under large-scale yielding conditions.
Limitations of Two-Parameter Fracture Mechanics

Assume that the damage ahead of the crack tip is characterized by second-order local stresses, which are dependent of specimen size and geometry as opposed to the first-order terms. Consequently, the stress field is no longer singular as $r \rightarrow 0$, and the second-order term in the series of expansion is known in the literature as the T-stress for elastic behavior which accounts for effects of stress biaxiality. For elastic-plastic and fully plastic materials, the second-order term is also known as the J-Q approach [9]. In particular, O’Dowd and Shih [40, 41] can be consulted for obtaining details of the J-Q theory which describes the fundamentals that provide quantitative measures of the crack tip deformation. Nevertheless, the term Q accounts for plasticity in the triaxiality state crack tip stress field.
Limitations of Two-Parameter Fracture Mechanics

The characterization of crack tip stress and strain fields is fundamental to fracture mechanics. In elastic–plastic fracture mechanics, it has now been well established the crack tip stress/strain fields in structural components show wide range of variations, and two parameter descriptions have been developed to characterize these stress/strain fields. In this methodology, the first parameter measures the degree of crack-tip deformation, as characterized by J (or equivalently CTOD). The second parameter, characterizes the degree of crack tip constraint, which quantifies the level of deviation of stress/strain fields from HRR fields. The most commonly used second parameters are T-stress, Q-factor and $A_2$-term, corresponding to the J–T, J–Q and J–$A_2$ characterizations. It has been showed that two-parameter approaches provide effective characterization of plane-strain elastic–plastic crack tip fields in a variety of crack configurations and loading conditions.
Limitations of Two-Parameter Fracture Mechanics

The figure shows the crack tip opening stress for a high constraint specimen geometry (such as bending or/and deeply cracked specimen) and low constraint specimen (such as shallow crack or/and under tension). In some cases of high constraint geometry, the stress distribution is close to the HRR even as the load becomes high. However, in the low constraint geometry the stresses gradually deviate from HRR solution as the load increases. In other words, under low loading or small scale yielding (SSY), the HRR solution could be used to characterize the stress fields. Under higher load or large scale yielding (LSY), the HRR solution may not be sufficient to characterize the crack tip stress field, especially for low constraint specimens.
Limitations of Two-Parameter Fracture Mechanics

- Low-constraint configurations like the center-cracked panel and shallow notched bend specimens diverge from single-parameter theory almost immediately.
- Deeply notched bend specimens maintain high constraint to relatively high J values.
- Low-constraint geometries can be treated with two-parameter theory, and high constraint geometries can be treated with single-parameter theory in many cases. When high constraint geometries violate the single-parameter assumption, however, two-parameter theory is of little value.
Governing fracture mechanism and fracture toughness
Fracture vs. Plastic collapse

\[
\sigma_{\text{net}} = \frac{P}{W - a} = \sigma \frac{W}{W - a} \quad \sigma = \frac{P}{W}
\]

(cracked section)

Yield:
\[
\sigma \frac{W}{W - a} = \sigma_{ys} \quad \rightarrow \quad \sigma = \sigma_{ys} \left(1 - \frac{a}{W}\right)
\]

**short crack: fracture by plastic collapse!!!**

**high toughness materials:** yielding before fracture

LEFM applies when \( \sigma_c \leq 0.66\sigma_{ys} \)
Fracture vs. Plastic collapse

Example: Estimate the failure load under uniaxial tension for a center-cracked panel of aluminum alloy of width $W=500$ mm, and thickness $B=4$ mm, for the following values of crack length $2a = 20$ mm and $2a = 100$ mm. Yield stress $\sigma_y = 350$ MPa and fracture toughness $K_{ic} = 70$ MPa $m$.

Solution: There are two possible failure modes: plastic collapse and brittle fracture. We will assess the load level required for each mode to prevail.

(i) $2a = 20$ mm. Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B = 672$ kN

Fracture load $F_c = \sigma_c \cdot W \cdot B$ where $\sigma_c = \frac{K_{ic}}{\sqrt{\pi a \sec(\pi a / W)}} = 394.6$ MPa

thus $F_c = 790$ kN.

The actual failure load is the smaller of the above results, 672 kN.

(ii) $2a = 100$ mm. Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B = 560$ kN

Fracture load $F_c = \sigma_c \cdot W \cdot B$ where $\sigma_c = \frac{K_{ic}}{\sqrt{\pi a \sec(\pi a / W)}} = 172.2$ MPa

thus $F_c = 334.57$ kN.

The actual failure load is the smaller of the above results, 334.57 kN.