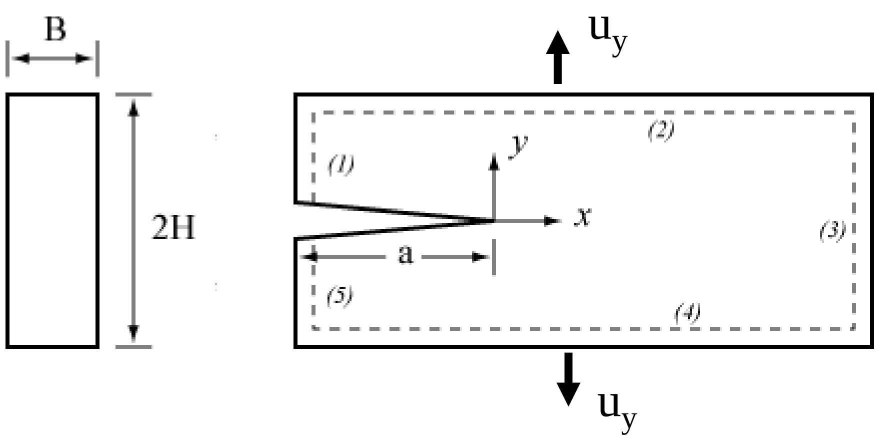
دانشگاه صنعتی اصفهان

دانشکده مکانیک

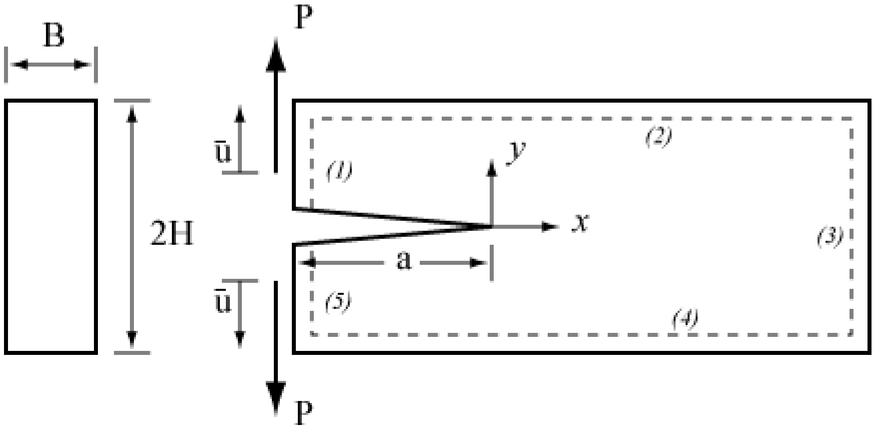
The J integral

Example 1



$$J = 2hw = \frac{(1-\nu)Eu_y^2}{(1+\nu)(1-2\nu)h}$$

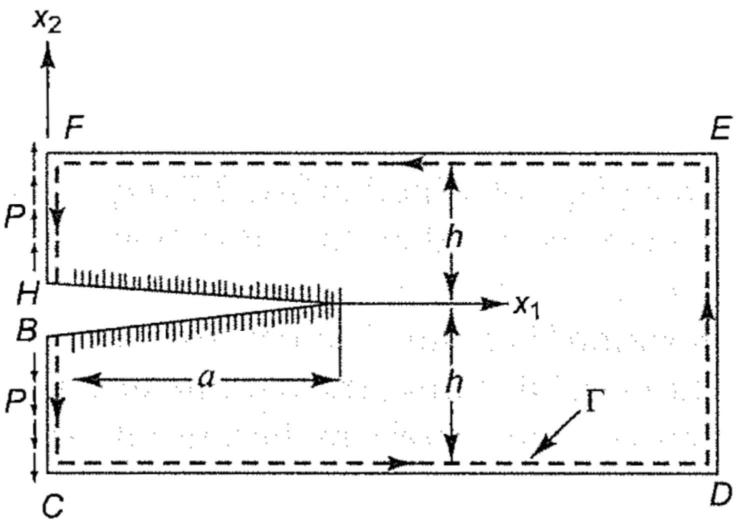
Example 2

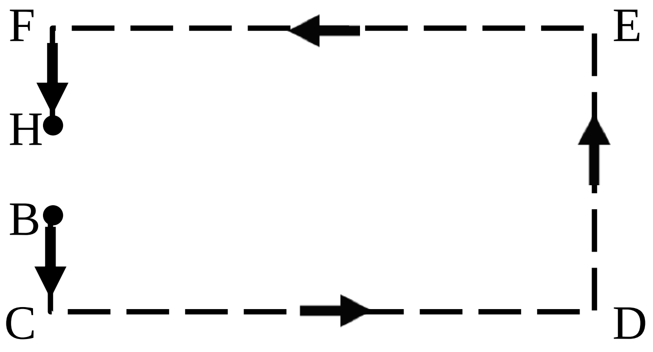


$$J = \frac{12P^2a^2}{EB^2h^3}$$

Example 3:

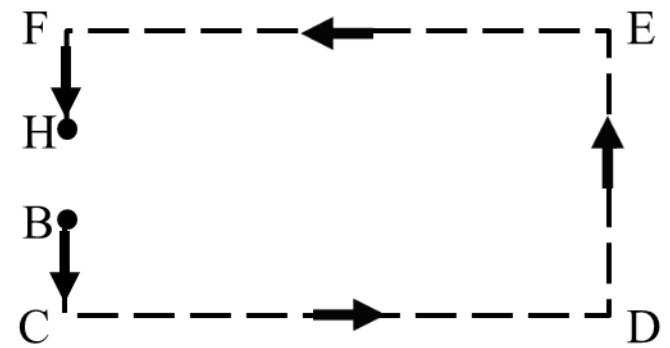
J integral for double cantilever beam, if each cantilever is pulled by a distributed load P, as shown





The chosen path Γ is BCDEFH and it coincides with the body contour.

Applications the J-integral



Contour of the crack; therefore, $J = J_{BC} + J_{CD} + J_{DE} + J_{EF} + J_{FH}$

J_{BC} : As bending moment is zero, bending stress is zero. So, $w=0$

$$J_{BC} = \int_{BC} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

$$= \int_{BC} \left(0 - T_i \frac{\partial u_i}{\partial x} ds \right)$$

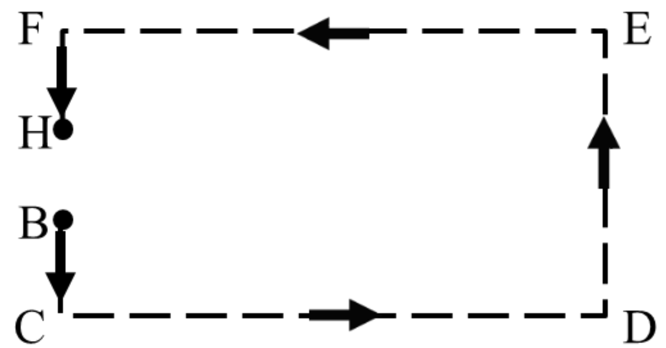
$$= - \int_{BC} T_i \frac{\partial u_i}{\partial x} ds$$

$$J_{BC} = - \int_0^h T \frac{\partial v}{\partial x} dy$$

Assuming:

- A small element of length dy on the path then $ds = dy$
- Along y-direction $u_i = v$
- Length is h so limits are 0 to h.

Applications the J-integral



Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$

J_{CD} :
$$J_{CD} = \int_{CD} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

dy is negligible and $T_i = 0$ ⇨ $J_{CD} = 0$

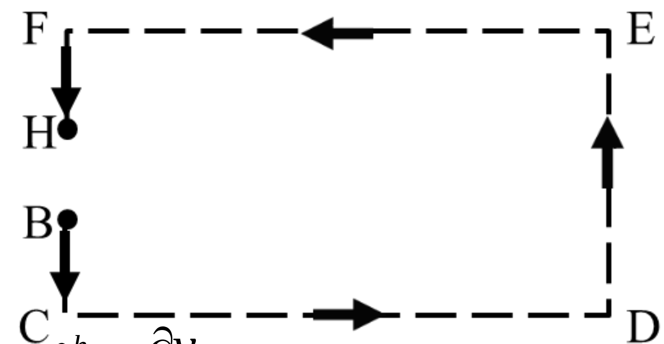
J_{EF} :
$$J_{EF} = \int_{EF} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

dy is negligible and $T_i = 0$ ⇨ $J_{EF} = 0$

J_{DF} :
$$J_{DE} = \int_{DE} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

stresses are very small, which in turn, make w and T_i negligible. ⇨ $J_{DE} = 0$

Applications the J-integral



Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$

J_{FH} :

$$J_{FH} = \int_{FH} \left(w\, dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

On segments BC and FH, w is negligible, ⟹ $J_{FH} = -\int_0^h T \frac{\partial v}{\partial x} dy$

Hence

$$J = J_{BC} + J_{CD} + J_{DE} + J_{EF} + J_{FH}$$

$$J = -\int_0^h T \frac{\partial v}{\partial x} dy + 0 + 0 + 0 - \int_0^h T \frac{\partial v}{\partial x} dy$$

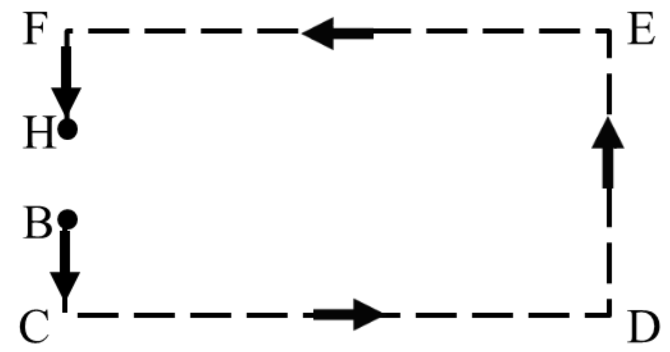
$$= -2\int_0^h T \frac{\partial v}{\partial x} dy$$

Now we can find $\frac{\partial v}{\partial x}$ using the bending moment equation; Bending moment = P*x

$$\frac{\partial^2 v}{\partial x^2} = \frac{Px}{EI} \Rightarrow \frac{\partial v}{\partial x} = \frac{P}{EI}\frac{x^2}{2} + c \quad \left(\text{at } x=a, \frac{\partial v}{\partial x} = 0\right) \Rightarrow c = -\frac{P}{EI}\frac{a^2}{2} \Rightarrow \frac{\partial v}{\partial x} = \frac{P}{EI}\frac{x^2}{2} - \frac{P}{EI}\frac{a^2}{2}$$

Applications the J-integral

Contour of the crack; therefore, $J=J_{BC}+J_{CD}+J_{DE}+J_{EF}+J_{FH}$



(at $x=0$, $\frac{\partial v}{\partial x} = -\frac{P}{EI}\frac{a^2}{2}$) and $I = \frac{Bh^3}{12}$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{6Pa^2}{EBh^3}$$

Hence
$$J = -2\int_0^h T^*\left(-\frac{6Pa^2}{EBh^3}\right)dy$$
$$= \frac{12Pa^2}{EBh^3}\int_0^h T dy$$

But on face FH: $B\int_0^h T dy = P \quad \Rightarrow \quad J = \frac{12P^2a^2}{EB^2h^3}$

Nonlinear Energy Release Rate

• Similar to linear elastic materials, the energy release rate defines for nonlinear elastic materials, except that G is replaced by J:

$$J = -\frac{d\Pi}{dA}$$

The Π is the potential energy and A is crack area: $\Pi = U - F$

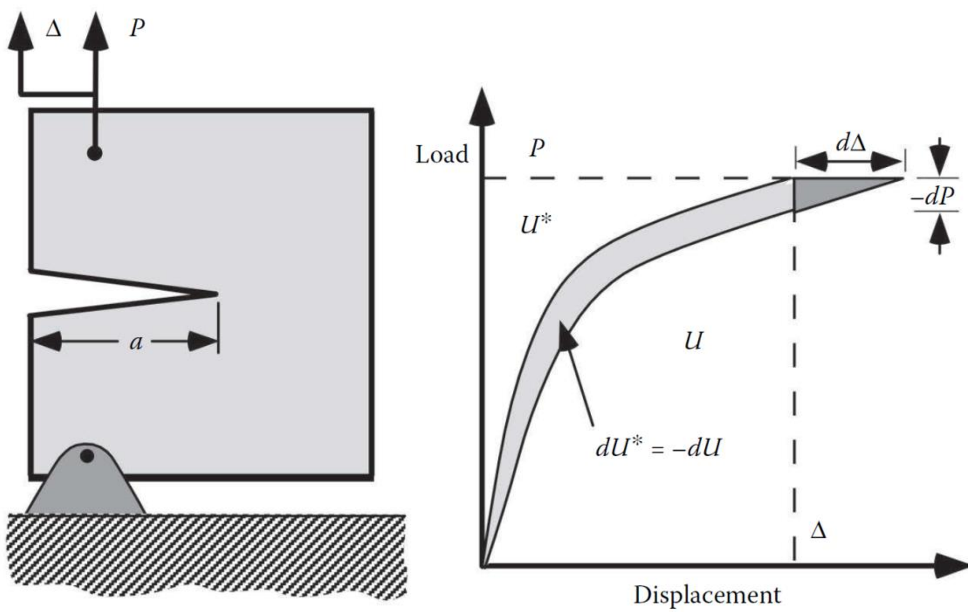
U is the strain energy stored in the body and F is the work done by external forces.

Consider a cracked plate which exhibits a nonlinear load–displacement curve, If the plate has unit thickness, A = a,

For load control: $\Pi = U - P\Delta = -U^*$

The U* is the complementary strain energy,:

$$U^* = \int_0^P \Delta dP$$



In the load control: $$J=\left(\frac{dU^*}{da}\right)_P$$

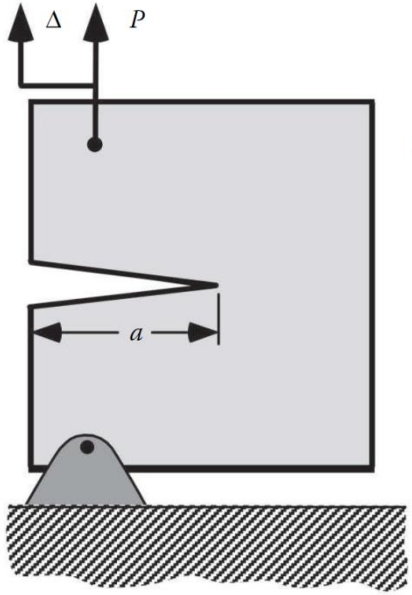
If the crack advances at a fixed displacement, $F = 0$, and J is given by: $$J=-\left(\frac{dU}{da}\right)_\Delta$$

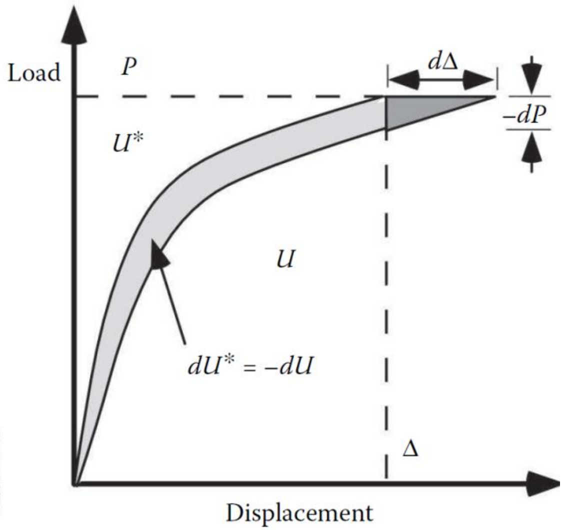
dU* for load control differs from −dU for displacement control by the amount ½dPΔ, which is vanishingly small compared to dU. Therefore, J for load control is equal to J for displacement control.

- The J in terms of load and displacement:

$$J=\left(\frac{\partial}{\partial a}\int_0^P \Delta dP\right)_P=\int_0^P\left(\frac{\partial \Delta}{\partial a}\right)_P dP$$

$$J=-\left(\frac{\partial}{\partial a}\int_0^\Delta P d\Delta\right)_\Delta=-\int_0^\Delta\left(\frac{\partial P}{\partial a}\right)_\Delta d\Delta$$



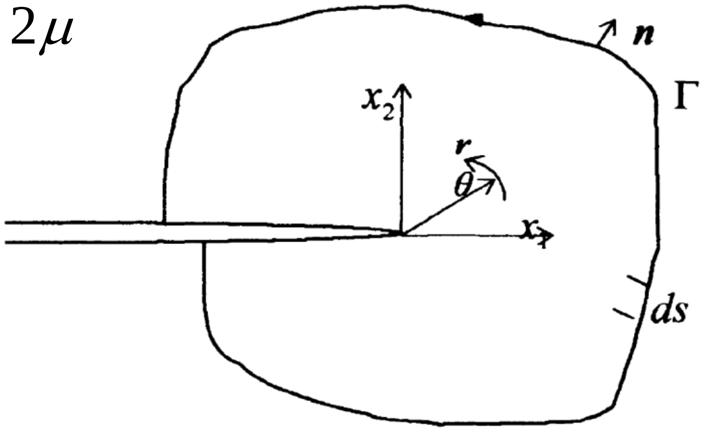


Example 4:

Write the general expression for the J-integral for anti-plane shear, i.e. mode III. Using the dominant elastic singularity, shrink the J-contour down to the crack tip and prove, by direct calculation, that

$$J = \frac{K_{III}^2}{2\mu}$$

$$J = \int_{\Gamma} \left(w\, n_1 - T_i \frac{\partial u_i}{\partial x_1} \right) ds$$



Path for the calculation of J-integral

where Γ is the path along which the J-integral is calculated, $w = \sigma_{ij}\varepsilon_{ij}/2$ is the strain energy density for linear elastic material, n_1 is the component of the outward unit normal **n** in the x_1-direction, $T_i = \sigma_{ij} n_j$ and $\mathbf{u}_i$ are, respectively, the traction and displacement vectors along Γ and *ds* is differential arc length.

Example 4:

For a crack under conditions of mode III deformation:

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{Bmatrix}, \qquad u_3 = \frac{2K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}$$

For mode III deformation, the strain energy density is:

$$w = \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23} = \sigma_{13}\frac{\sigma_{13}}{2\mu} + \sigma_{23}\frac{\sigma_{13}}{2\mu} = \frac{1}{2\mu}\left(\sigma_{13}^2 + \sigma_{23}^2\right) = \frac{1}{2\mu}\left(\frac{K_{III}^2}{2\pi r}\right) = \frac{K_{III}^2}{4\pi\mu r}$$

Traction: $T_1 = T_2 = 0$

$$T_3 = \sigma_{31}n_1 + \sigma_{32}n_2 = \sigma_{31}\cos\theta + \sigma_{32}\sin\theta$$

$$= \frac{K_{III}}{\sqrt{2\pi r}}\left(-\sin\frac{\theta}{2}\cos\theta + \cos\frac{\theta}{2}\sin\theta\right) = \frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

Example 4:

Displacement Derivative: Since the displacement u_3 is a function of the polar coordinates r and θ

$$\frac{\partial u_3}{\partial x_1}=\frac{\partial u_3}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial u_3}{\partial \theta}\frac{\partial \theta}{\partial x_1}=(\frac{K_{III}}{\mu\sqrt{2\pi r}}\sin\frac{\theta}{2})(\cos\theta)+(\frac{K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2})(-\frac{1}{r}\sin\theta)$$

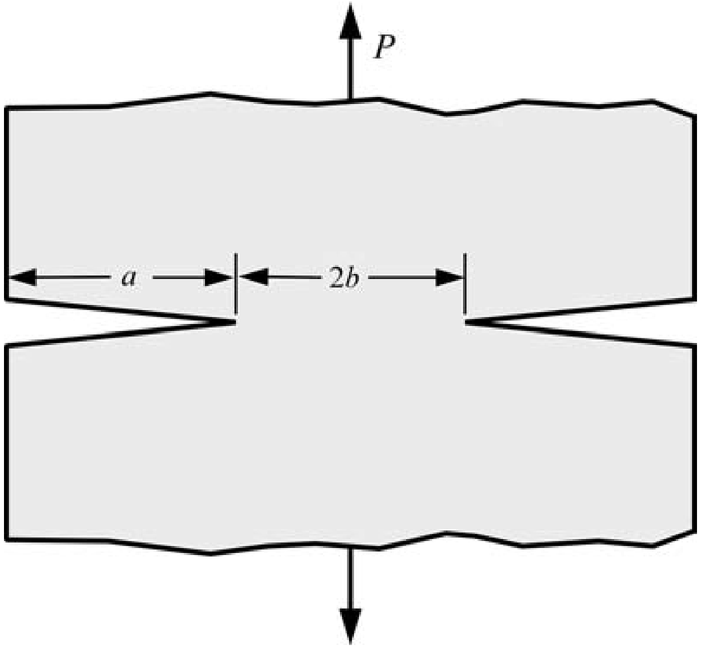
$$=-\frac{K_{III}}{\mu\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

The J-integra1 for mode III deformation is:

$$J=\frac{K_{III}^2}{2\pi\mu r}\int_{-\pi}^{\pi}\left(\frac{1}{2}\cos\theta+\sin^2\frac{\theta}{2}\right)rd\theta=\frac{K_{III}^2}{2\pi\mu r}(\frac{1}{2}\theta\Big|_{-\pi}^{\pi})(r)=\frac{K_{III}^2}{2\pi\mu r}\frac{1}{2}2\pi r=\frac{K_{III}^2}{2\mu}$$

Applications the J-integral

Double-edge-notched tension (DENT)



$$J = \frac{1}{2}\int_0^P \left(\frac{\partial \Delta}{\partial a}\right)_P dP = -\frac{1}{2}\int_0^P \left(\frac{\partial \Delta}{\partial b}\right)_P dP$$

$$\Delta = \Delta_{el} + \Delta_p$$

$$J = -\frac{1}{2}\int_0^P \left[\left(\frac{\partial \Delta_{el}}{\partial b}\right)_P + \left(\frac{\partial \Delta_p}{\partial b}\right)_P\right] dP$$

$$= \frac{K_I^2}{E'} - \frac{1}{2}\int_0^P \left(\frac{\partial \Delta_p}{\partial b}\right)_P dP$$

Assume that b is the only length dimension that influences Δ_p.

For plane stress

$E' = E$

For plane strain

$E' = E/(1-\nu^2)$

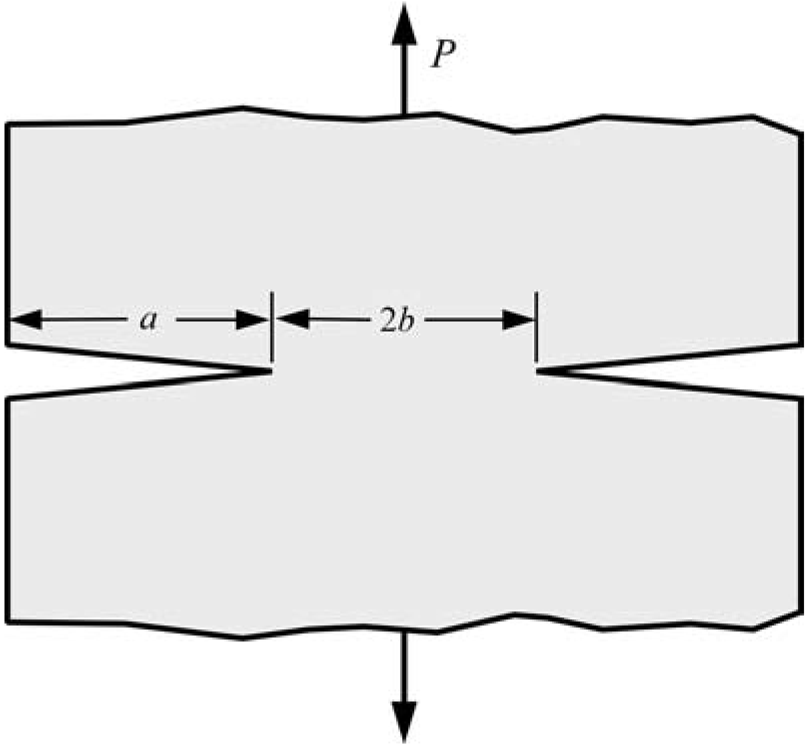
$$\Delta_p = bH\left(\frac{P}{b}\right) \Rightarrow \left(\frac{\partial \Delta_p}{\partial b}\right)_P = H\left(\frac{P}{b}\right) - H'\left(\frac{P}{b}\right)\frac{P}{b}$$

$$\left(\frac{\partial \Delta_p}{\partial P}\right)_b = H'\left(\frac{P}{b}\right)$$

$$\left(\frac{\partial \Delta_p}{\partial b}\right)_P = \frac{1}{b}\left[\Delta_p - P\left(\frac{\partial \Delta_p}{\partial P}\right)_b\right]$$

Double-edge-notched tension (DENT)

$$\left(\frac{\partial \Delta_p}{\partial b}\right)_P = \frac{1}{b}\left[\Delta_p - P\left(\frac{\partial \Delta_p}{\partial P}\right)_b\right]$$



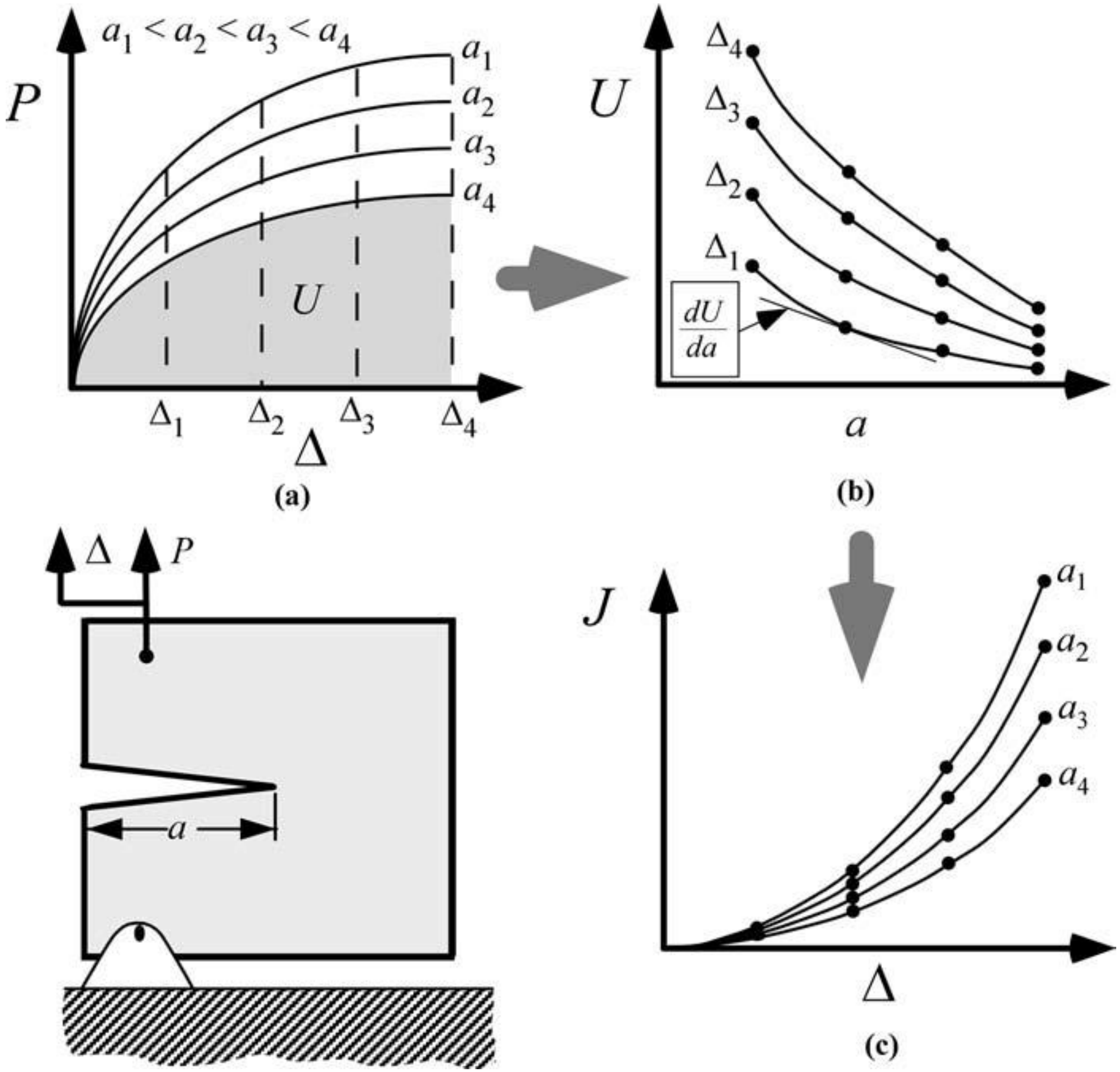
$$J = -\frac{1}{2}\int_0^P \left[\left(\frac{\partial \Delta_{el}}{\partial b}\right)_P + \left(\frac{\partial \Delta_p}{\partial b}\right)_P\right] dP$$

$$= \frac{K_I^2}{E'} - \frac{1}{2}\int_0^P \left(\frac{\partial \Delta_p}{\partial b}\right)_P dP$$

$$J = \frac{K_I^2}{E'} + \frac{1}{2b}\left[2\int_0^{\Delta_p} P\,d\Delta_p - P\Delta_p\right]$$

Laboratory Measurement

Laboratory Measurement



Computing the J integral is somewhat difficult when the material is nonlinear

U vs. crack length
at various fixed displacements

$$J = -\frac{1}{B}\left(\frac{\partial U}{\partial a}\right)_{\Delta}$$