دانشگاه صنعتی اصفهان
دانشکده مکانیک

The J integral

The J contour integral as yield criterion

- J as a path-independent line integral

$$J = \int_\Gamma \left(w\,dy - T_i \frac{\partial u_i}{\partial x} ds \right) \quad \text{with} \quad w(\varepsilon_{mn}) = \int_0^{\varepsilon_{mn}} \sigma_{ij}\, d\varepsilon_{ij} \quad \text{strain energy density}$$

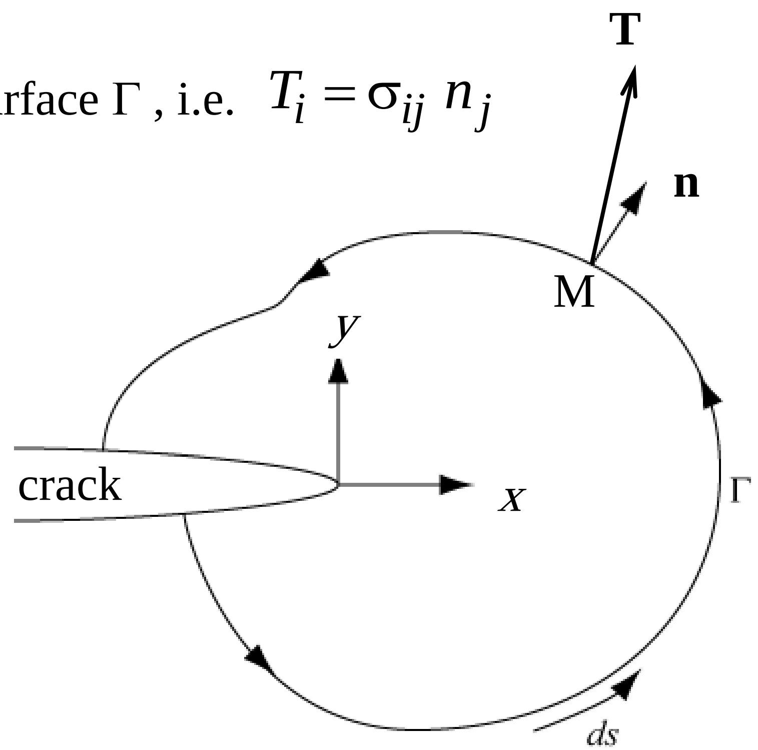
$$= \int_\Gamma \left(w\,dy - \mathbf{T}\cdot \frac{\partial \mathbf{u}}{\partial x} ds \right)$$

T : traction vector at a point M on the bounding surface Γ , i.e. $T_i = \sigma_{ij}\, n_j$

u : displacement vector at the same point M.

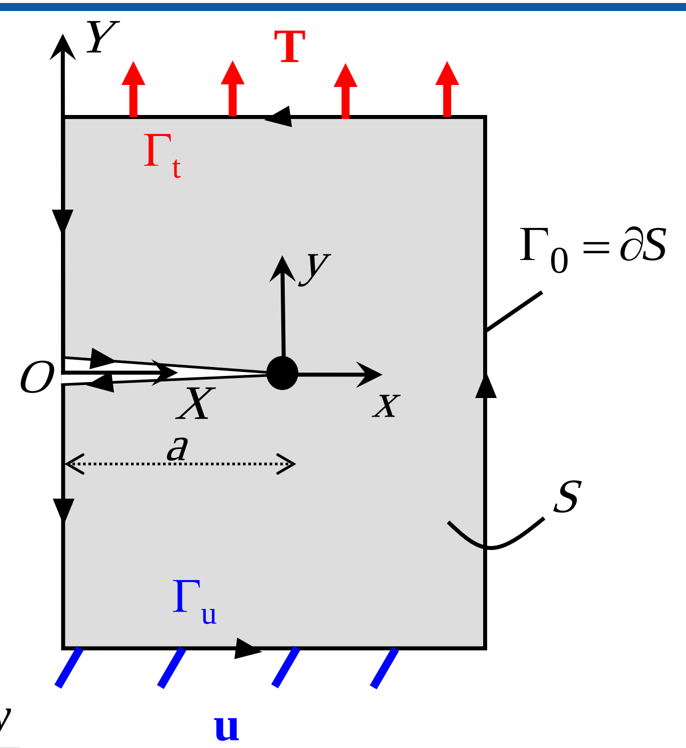
n : unit *outward* normal.

The contour Γ is followed in the *counter-clockwise* direction.



Equivalence of the two definitions

- 2D solid of unit thickness of area S ,
  with a linear crack of length a along OX (fixed)
- Crack faces are traction-free.
- Total contour of the solid Γ_0 *including* the crack tip:
  Imposed tractions on the part of the contour Γ_t
  Displacements applied on Γ_u



Proof : Recall for the potential energy (per unit thickness),

$$\Pi(a) = \iint_S w\,dS - \int_{\Gamma_t} T_i u_i \, ds \qquad T_i = \sigma_{ij} n_j \qquad \sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$$

The tractions and displacements imposed on Γ_t and Γ_u are independent of a

$$\frac{dT_i}{da} = 0, \quad on \quad \Gamma_t$$

$$\frac{du_i}{da} = 0 \quad on \quad \Gamma_u$$

$$\Rightarrow \quad \frac{d\Pi}{da} = \iint_S \frac{dw}{da} dS - \int_{\Gamma_0} T_i \frac{du_i}{da} ds$$

The J contour integral as yield criterion

Considering the moving coordinate system $x\,,\,y$ (attached to the crack tip), $x = X - a$

$\frac{d}{da}$: total derivative/crack length

$$\frac{d}{da} = \left(\frac{\partial}{\partial a}\right)_x + \left(\frac{\partial x}{\partial a}\right)_X \left(\frac{\partial}{\partial x}\right)_a \qquad = \frac{\partial}{\partial a} - \frac{\partial}{\partial x}$$

Thus,

$$\frac{d\Pi}{da} = \iint\limits_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int\limits_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

However,

$$\frac{\partial w}{\partial a} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial a} \quad = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial a} \quad = \sigma_{ij} \frac{\partial}{\partial a}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

$$= \sigma_{ij} \frac{\partial}{\partial a} \frac{\partial u_i}{\partial x_j} \quad = \quad \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right)$$

Thus,

$$\iint_S \frac{\partial w}{\partial a} dS = \iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS$$

From the divergence theorem:

$$\iint_S \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial a}\right) dS = \int_{\Gamma_0} \sigma_{ij} \frac{\partial u_i}{\partial a} n_j \, ds = \int_{\Gamma_0} T_i \frac{\partial u_i}{\partial a} \, ds$$

The derivative of J reduces to,

$$\frac{d\Pi}{da} = \iint_S \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\iint_S \left(\frac{\partial w}{\partial x}\right) dS + \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a}\right) ds - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x}\right) ds$$

$$= -\left(\iint_S \left(\frac{\partial w}{\partial x}\right) dS - \int_{\Gamma_0} T_i \left(\frac{\partial u_i}{\partial x}\right) ds\right)$$

The J contour integral as yield criterion

Using the Green Theorem, i.e. $$\oint_{\Gamma} P(x,y)dx + Q(x,y)dy = \iint_{A}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$-\frac{d\Pi}{da} = \iint_{S}\left(\frac{\partial w}{\partial x}\right)dS - \int_{\Gamma_0} T_i\left(\frac{\partial u_i}{\partial x}\right)ds$$

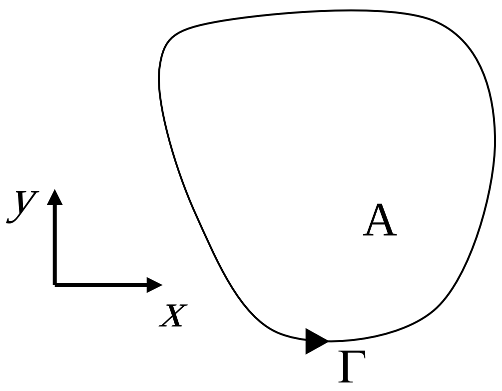
$$= \int_{\Gamma_0}\left(w\,dy - T_i\left(\frac{\partial u_i}{\partial x}\right)ds\right)$$

➡ J derives from a potential

Properties of the J-integral

1) J is *zero* for any closed contour containing *no crack tip.*

Closed contour around A



Consider $J\big|_{\Gamma} = \oint_{\Gamma} \left(w\,dy - T_i \frac{\partial u_i}{\partial x} ds \right)$

Using the Green Theorem, i.e. $\oint_{\Gamma} P(x,y)\,dx + Q(x,y)\,dy = \iint_{A} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

We have $J\big|_{\Gamma} = \int_{A} \frac{\partial w}{\partial x} dx\,dy - \oint_{\Gamma} T_i \frac{\partial u_i}{\partial x} ds \quad = \int_{A} \frac{\partial w}{\partial x} dx\,dy - \oint_{\Gamma} \sigma_{ij} \frac{\partial u_i}{\partial x} n_j\, ds$

From the divergence theorem,

$$\oint_{\Gamma} \sigma_{ij} \frac{\partial u_i}{\partial x} n_j\, ds \quad = \int_{A} \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) dx\,dy$$

The J contour integral as yield criterion

The integral becomes,

$$J\big|_{\Gamma} = \int_A \left[\frac{\partial w}{\partial x} - \frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) \right] dxdy$$

However,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \qquad \text{since} \quad \sigma_{ij} = \sigma_{ji}$$

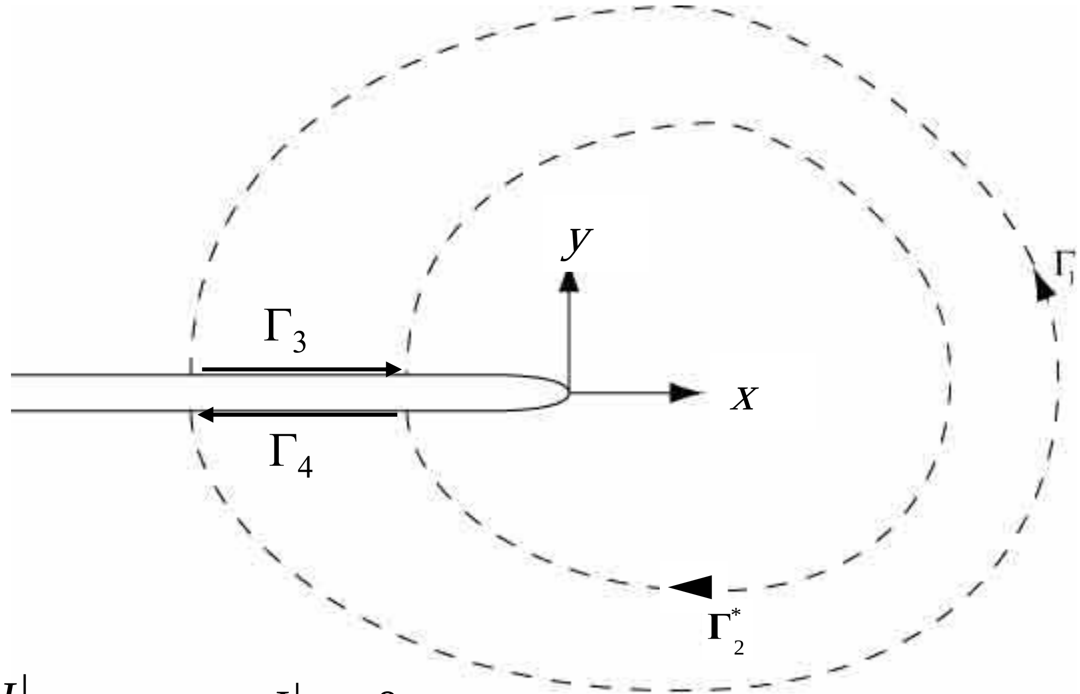
$$= \sigma_{ij} \frac{\partial}{\partial x} \frac{\partial u_i}{\partial x_j} = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Invoking the equilibrium equation, $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x} \right) = \underbrace{\frac{\partial \sigma_{ij}}{\partial x_j}}_{0} \frac{\partial u_i}{\partial x} + \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right) = \sigma_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x} \right)$$

Replacing in the integral, $J\big|_{\Gamma} = 0$

2) J is path-independent



Consider the *closed* contour:

$$\Gamma = \Gamma_1 + \Gamma_3 + \Gamma^*_2 + \Gamma_4$$

We have $J|_{\Gamma} = J|_{\Gamma_1} + J|_{\Gamma^*_2} + J|_{\Gamma_3} + J|_{\Gamma_4}$ and $J|_{\Gamma} = 0$

The crack faces are traction free :

$T_i = \sigma_{ij}\, n_j = 0$ on Γ_3 and Γ_4

$dy = 0$ along these contours

$\Rightarrow \quad J|_{\Gamma_3} = J|_{\Gamma_4} = 0$

The J contour integral as yield criterion

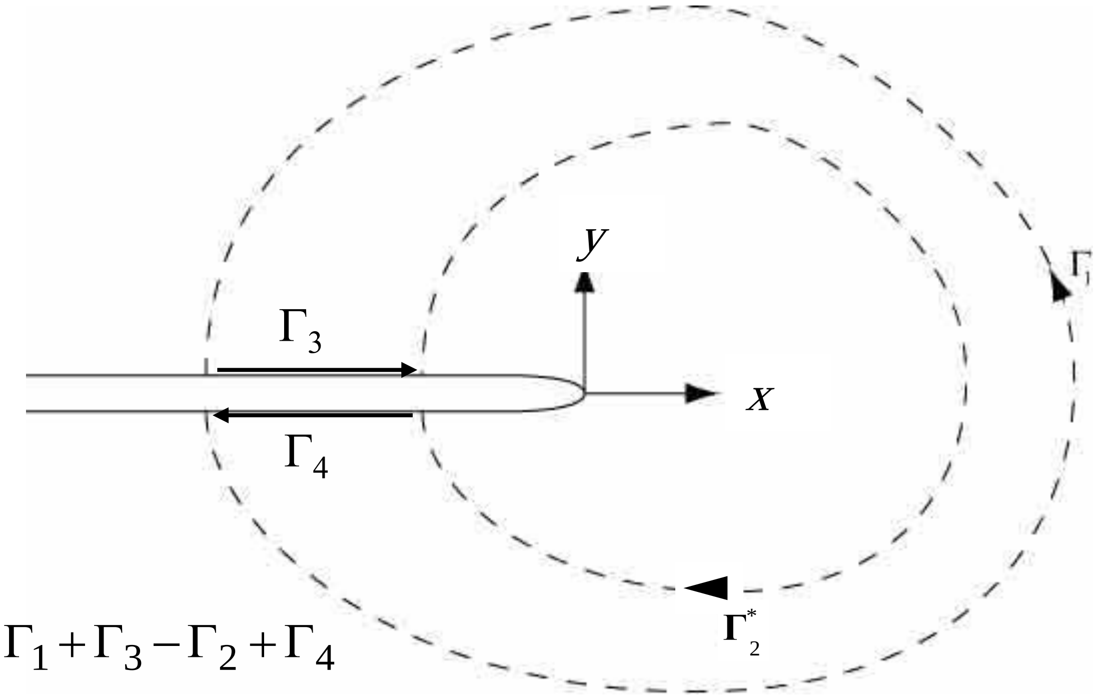
2) J is path-independent

Note that,

$$J\big|_{\Gamma_2^*} = -J\big|_{\Gamma_2}$$

and $\quad J\big|_{\Gamma} = J\big|_{\Gamma_1} - J\big|_{\Gamma_2} = 0$

$\Rightarrow \quad J\big|_{\Gamma_1} = J\big|_{\Gamma_2}$



$$\Gamma = \Gamma_1 + \Gamma_3 - \Gamma_2 + \Gamma_4$$

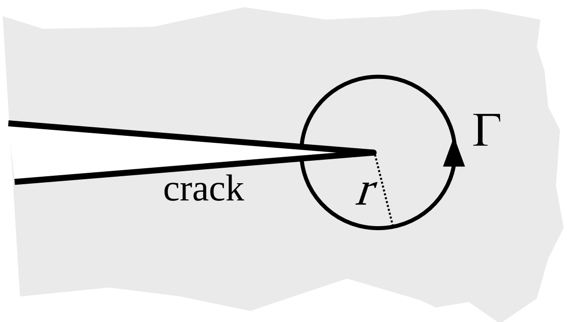
Γ_2 followed in the *counter-clockwise* direction.

Any arbitrary (counterclockwise) path around a crack gives the same value of J

$\Rightarrow$ J is *path*-independent

The J contour integral as yield criterion

J can be evaluated when the path is a circle of radius r around the crack tip



Γ is followed from $\theta = -\pi$ to $\theta = \pi$

We have,

$$ds = r d\theta$$
$$dy = r \cos\theta \, d\theta$$

J integral becomes,

$$J = \int_{-\pi}^{\pi} \left[w(r,\theta)\cos\theta - T_i(r,\theta)\,\frac{\partial u_i(r,\theta)}{\partial x} \right] r\, d\theta$$

When $r \to 0$ only the singular terms remain

For LEFM , we can obtain : $J = G = \frac{K_I^2}{E'}$ (if mode I loading)

HRR theory

Hutchinson **R**ice and **R**osengren: J characterizes the crack-tip field in a non-linear elastic material.

- For uniaxial deformation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$ Ramberg-Osgood equation

σ_0 = yield strength

$\varepsilon_0 = \sigma_0 / E$

α : dimensionless constant

n : strain-hardening exponent

(α and n: material properties)

Power law relationship assumed between plastic strain and stress.

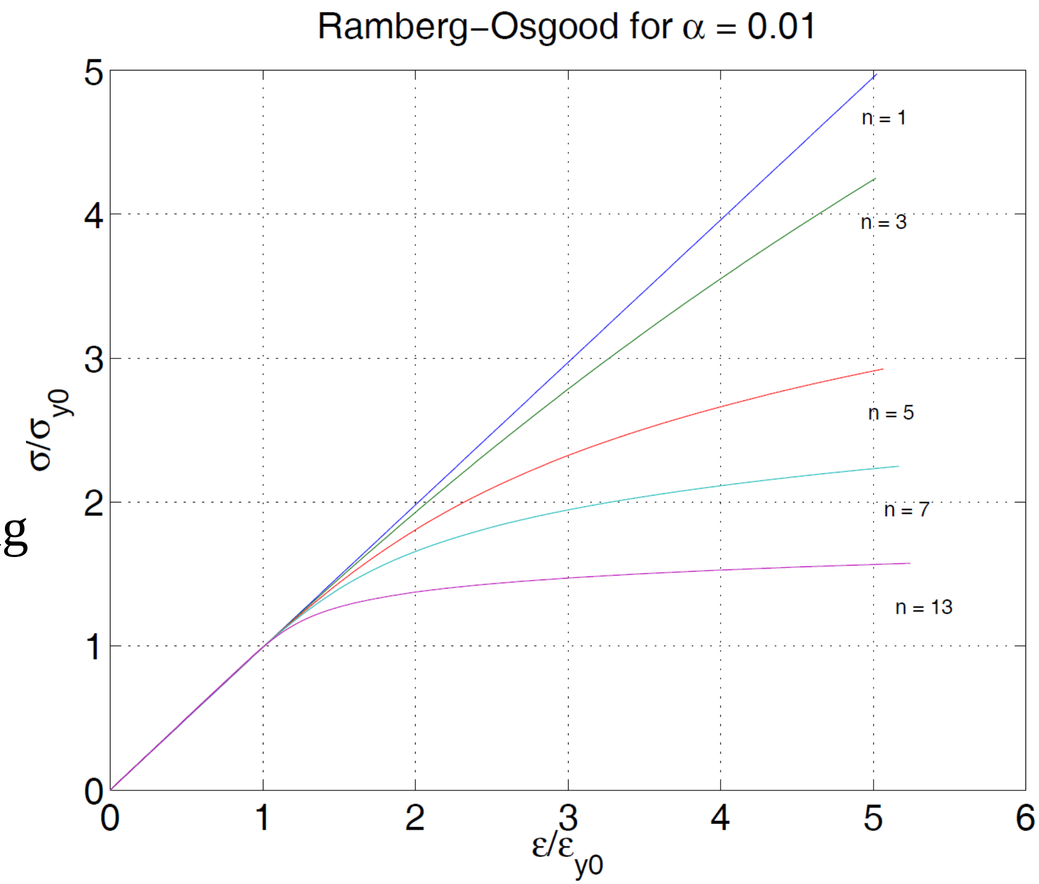
For a linear elastic material $n = 1$.

Ramberg–Osgood model

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

$$\varepsilon = \varepsilon^{el} + \varepsilon^{pl}$$

- Elastic model: Unlike plasticity unloading in on the same line
- Higher *n* closer to elastic perfectly plastic



Stress-strain relation according to the Ramberg-Osgood material law

The J contour integral as yield criterion

Hutchinson, Rice and Rosengren(HRR) solution

- Near crack tip "plastic" strains dominate:

$$\frac{\varepsilon}{\varepsilon_0} = \alpha\left(\frac{\sigma}{\sigma_0}\right)^n \qquad (*)$$

- Assume the following r dependence for σ and ε

$$\sigma = \frac{c_1}{r^x}$$

$$\varepsilon = \frac{c_2}{r^y}$$

1. Bounded energy:

$$\sigma\varepsilon \propto \frac{1}{r} \quad \Rightarrow \quad x + y = 1$$

2. $\varepsilon - \sigma$ relation (*)

$$y = nx$$

$$\Rightarrow \quad x = \frac{1}{1+n}, \qquad y = \frac{n}{1+n}$$

The J contour integral as yield criterion

- Asymptotic field derived by **H**utchinson **R**ice and **R**osengren:

$$\varepsilon_{ij} = A_2\left(\frac{J}{r}\right)^{n/(n+1)} \qquad \sigma_{ij} = A_1\left(\frac{J}{r}\right)^{1/(n+1)} \qquad u_i = A_3\, J^{\,n/(n+1)}\, r^{1/(n+1)}$$

A_i are regular functions that depend on θ and the previous parameters.

The $1/\sqrt{r}$ singularity is recovered when $n = 1$.

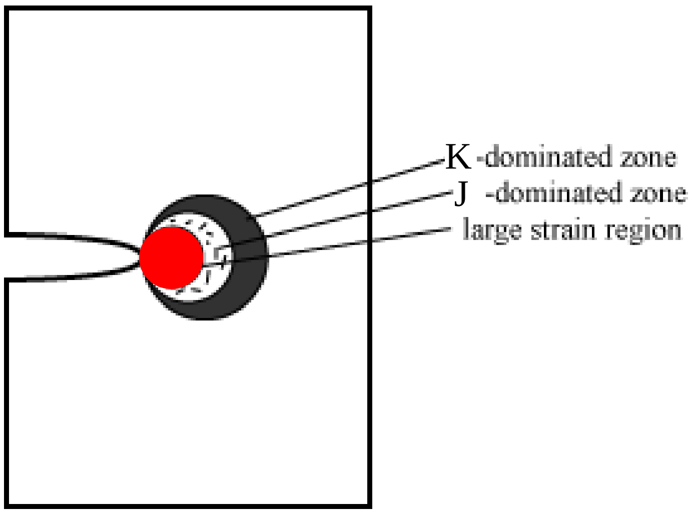
Path independence of J $\Rightarrow$ The product $\sigma_{ij}\,\varepsilon_{ij}$ varies as $1/r$:

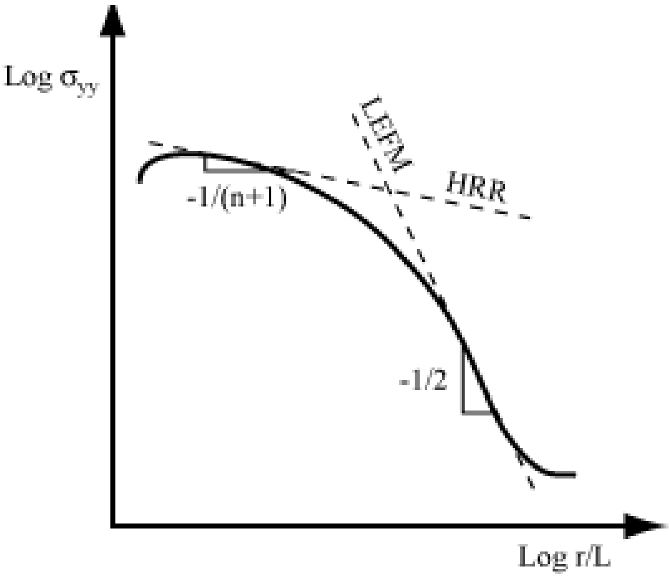
From $$J = r\int_{-\pi}^{\pi}\left[w(r,\theta)\cos\theta - T_i(r,\theta)\,\frac{\partial u_i(r,\theta)}{\partial x}\right]d\theta$$

$$\sigma_{ij}\,\varepsilon_{ij} \rightarrow \frac{f(\theta)}{r} \quad as \quad r \rightarrow 0$$

J defines the amplitude of the HRR field as K does in the linear case.

Two singular zones can be identified:





Small region where crack blunting occurs.

↳ Large deformation

HRR based upon small displacements non applicable.

The J contour integral as yield criterion

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^{\,2} I_n r} \right)^{1/(n+1)} \tilde{\sigma}_{ij}(n,\theta)$$

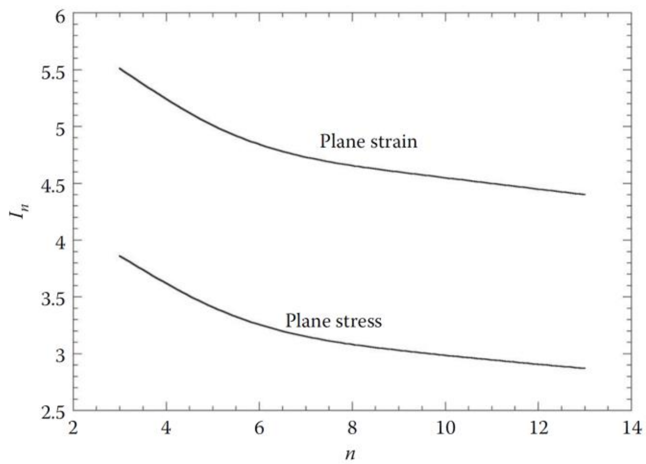
$$\varepsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^{\,2} I_n r} \right)^{n/(n+1)} \tilde{\varepsilon}_{ij}(n,\theta)$$

where I_n is an integration constant that depends on n, and $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are dimensionless functions of n and θ.

The equations are called the HRR singularity, named after Hutchinson, Rice, and Rosengren.

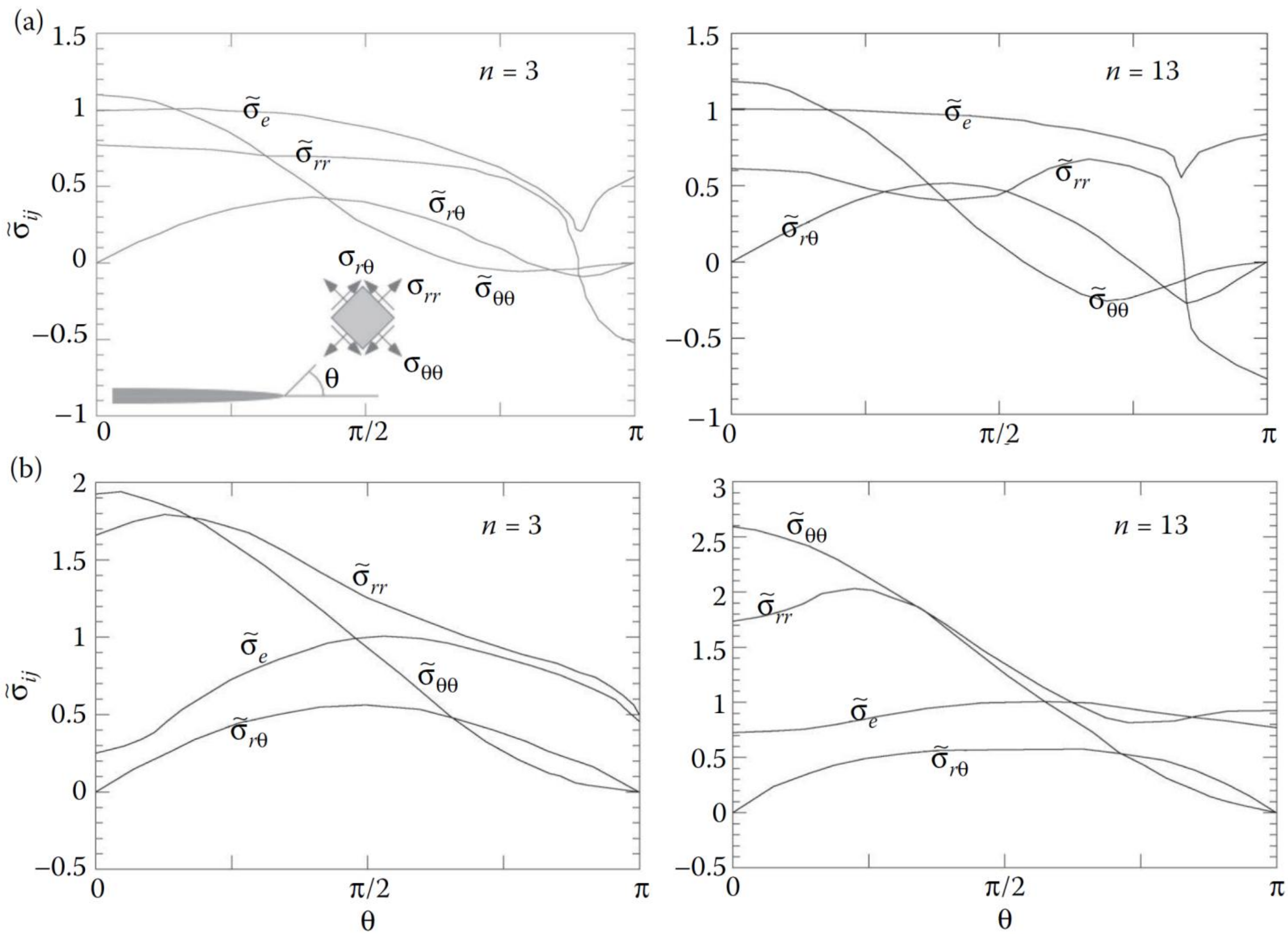
J defines the amplitude of the HRR field as K does in the linear case.

The J contour integral as yield criterion



Effect of the strain hardening exponent on the HRR integration constant

The J contour integral as yield criterion

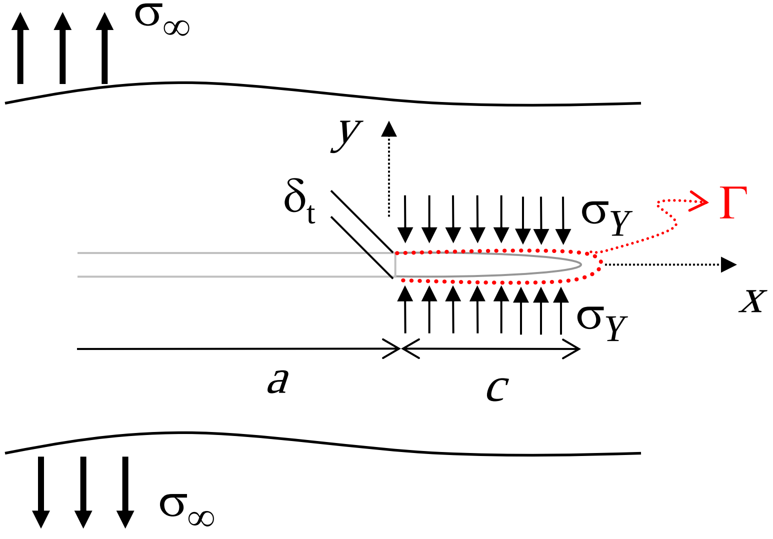


Angular variation of dimensionless stress for n = 3 and 13 (a) plane stress and (b) plane strain.

The J contour integral as yield criterion

- Relationship between J and CTOD

Consider again the strip-yield problem,



The first term in the J integral vanishes because *dy= 0* (slender zone)

$$J = -\int_{\Gamma} \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds$$

$$\text{but} \quad \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds = \sigma_{yy} n_y \frac{\partial u_y}{\partial x} ds = -\sigma_Y \frac{\partial u_y}{\partial x} dx$$

$$J = \int_{\Gamma} \sigma_Y \frac{\partial u_y}{\partial x} dx = \int_{-\delta_t}^{\delta_t} \sigma_Y du_y = \sigma_Y \delta_t$$

The J contour integral as yield criterion

General unique relationship between J and CTOD:

$$J = m\,\sigma_Y\,\delta_t$$

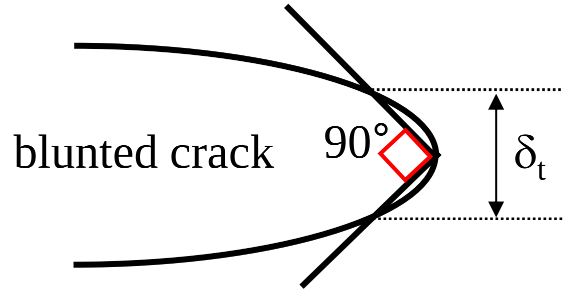
m: dimensionless parameter depending on the stress state and materials properties

- The strip-yield model predicts that m=1 (non-hardening material, plane stress condition)

- This relation is more generally derived for *hardening* materials (n >1) using the HRR displacements near the crack tip, i.e.

$$u_i = A_3\,J^{n/(n+1)}\,r^{1/(n+1)}$$

Shih proposed this definition for δ_t:



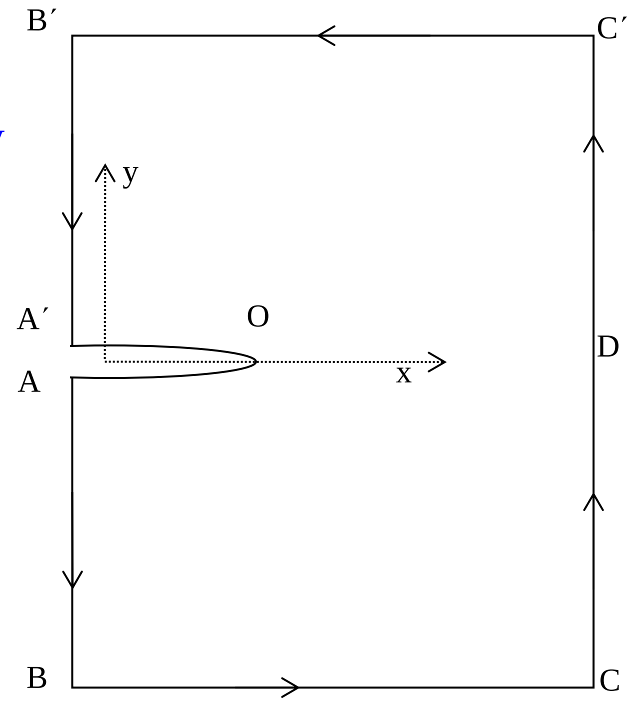
➡ m becomes a (complicated) function of n

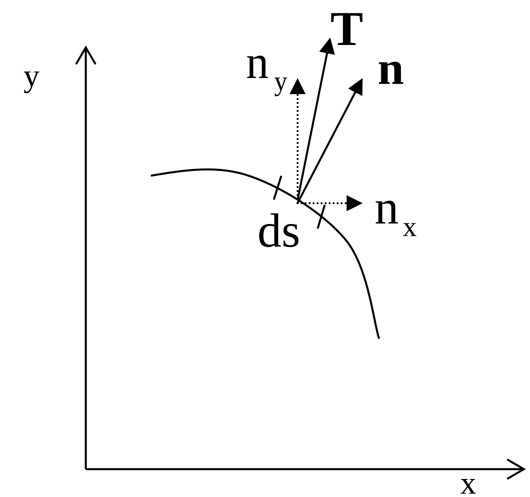
➡ The proposed definition of δ_t agrees with the one of the Irwin model

Moreover, $G = \frac{\pi}{4}\sigma_Y\delta_t$ $\quad m = \frac{\pi}{4}$ in this case

Applications the J-integral

J-integral evaluated explicitly along specific contours





Loads and geometry symmetric / Ox

$$J = \int_{\Gamma} \left(w\, dy - \sigma_{ij} n_j \frac{\partial u_i}{\partial x} ds \right) \quad ?$$

$$w = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + 2\sigma_{xy}\varepsilon_{xy}\right)$$

for a plane stress, linear elastic problem

The J contour integral as yield criterion

From stress-strain relation,

$$w=\frac{1}{2E}\left(\sigma_{xx}^{2}+\sigma_{yy}^{2}-2\nu\sigma_{xx}\sigma_{yy}\right)+\frac{1+\nu}{E}\sigma_{xy}^{2}$$

Expanded form for $\sigma_{ij}n_j\frac{\partial u_i}{\partial x}ds$

$$=\sigma_{xx}n_x\frac{\partial u_x}{\partial x}ds+\sigma_{xy}n_y\frac{\partial u_x}{\partial x}ds+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}ds+\sigma_{yy}n_y\frac{\partial u_y}{\partial x}ds$$ (2D problem)

Simplification :

Along AB or B´A´ $n_x = -1,\ n_y = 0$ and $ds=-dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}n_x\frac{\partial u_y}{\partial x}dy$$

Along CD or DC´ $n_x = 1,\ n_y = 0$ and $ds=dy \neq 0$

$$=\sigma_{xx}\frac{\partial u_x}{\partial x}dy+\sigma_{yx}\frac{\partial u_y}{\partial x}dx$$

The J contour integral as yield criterion

Along BC or C´B´

BC : $n_x = 0$, $n_y = -1$ and $ds=dx \neq 0$

$$=-\sigma_{xy}\frac{\partial u_x}{\partial x}dx-\sigma_{yy}\frac{\partial u_y}{\partial x}dx$$

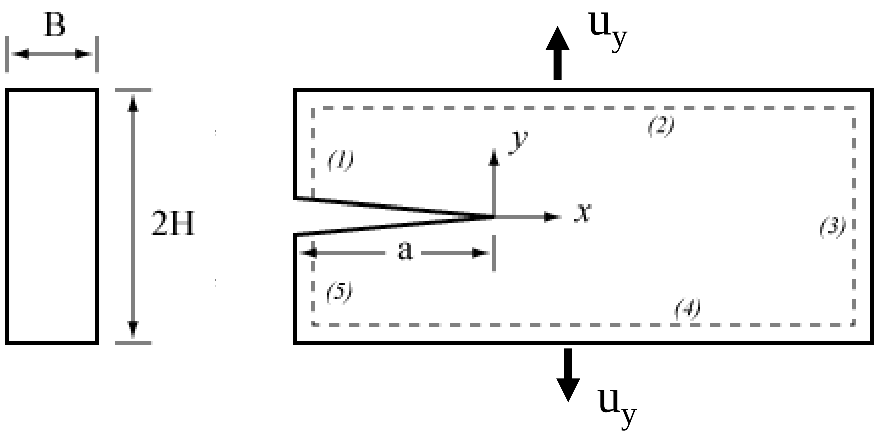
C'B' : $n_x = 0$, $n_y = 1$ and $ds=-dx \neq 0$

Along OA and A´O J is zero since $dy = 0$ and $T_i = 0$

Finally,

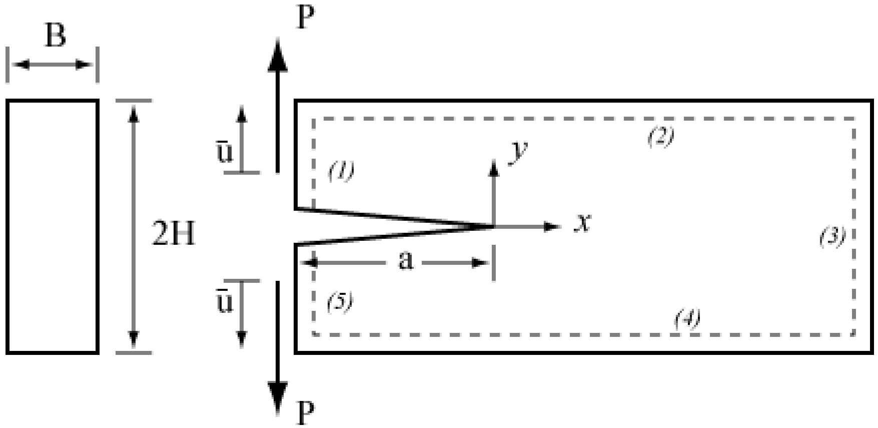
$$J=2\int_A^B\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy+2\int_B^C\left[\sigma_{xy}\frac{\partial u_x}{\partial x}+\sigma_{yy}\frac{\partial u_y}{\partial x}\right]dx+2\int_C^D\left[w-\sigma_{xx}\frac{\partial u_x}{\partial x}-\sigma_{xy}\frac{\partial u_y}{\partial x}\right]dy$$

Example 1



$$J=2hw=\frac{(1-\nu)Eu_y^2}{(1+\nu)(1-2\nu)h}$$

Example 2



$$J=\frac{12P^2a^2}{EB^2h^3}$$