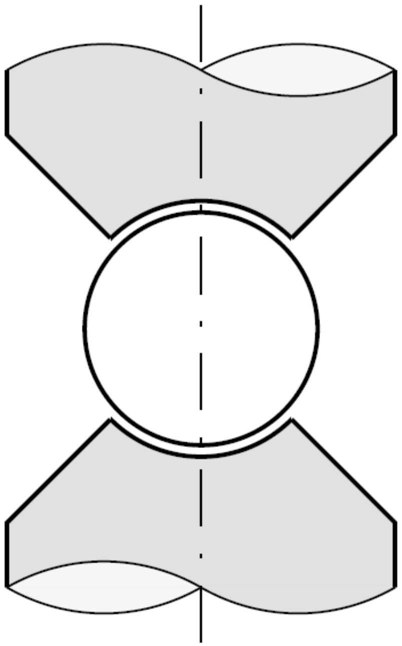
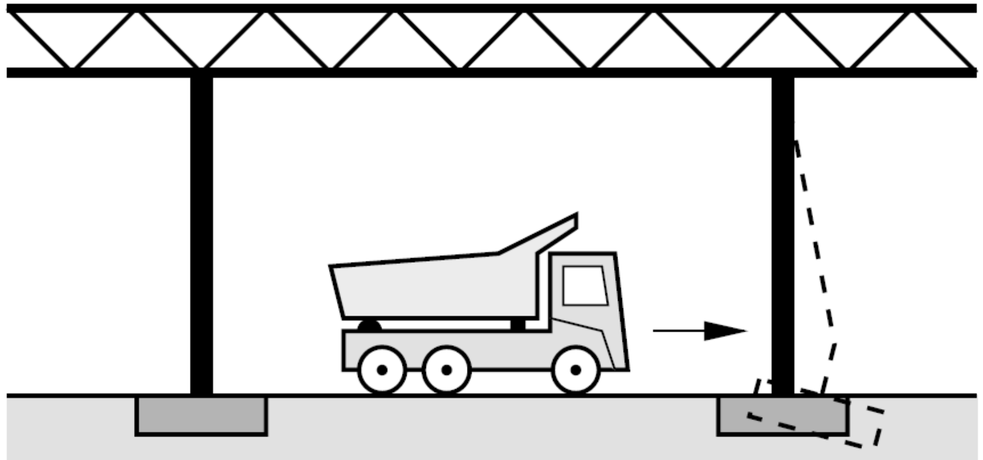
Introduction to Contact Mechanics

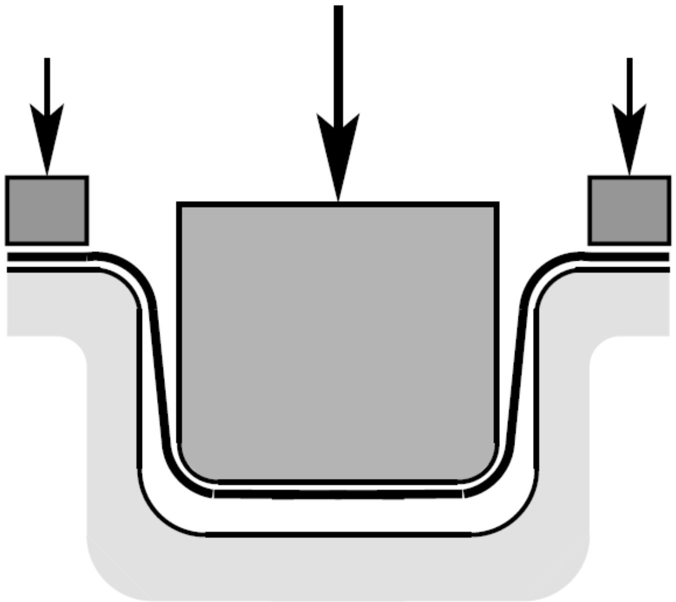
Contact problems: roller bearing and impact of a lorry.





Introduction to Contact Mechanics

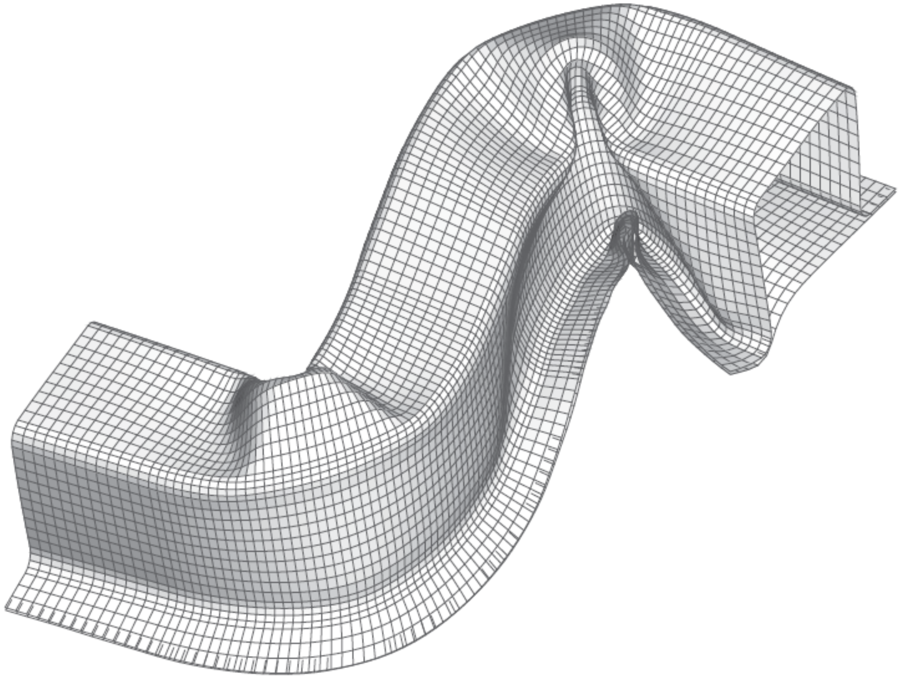
Sheet metal forming



Crash of a car against a deformable barrier

Introduction to Contact Mechanics

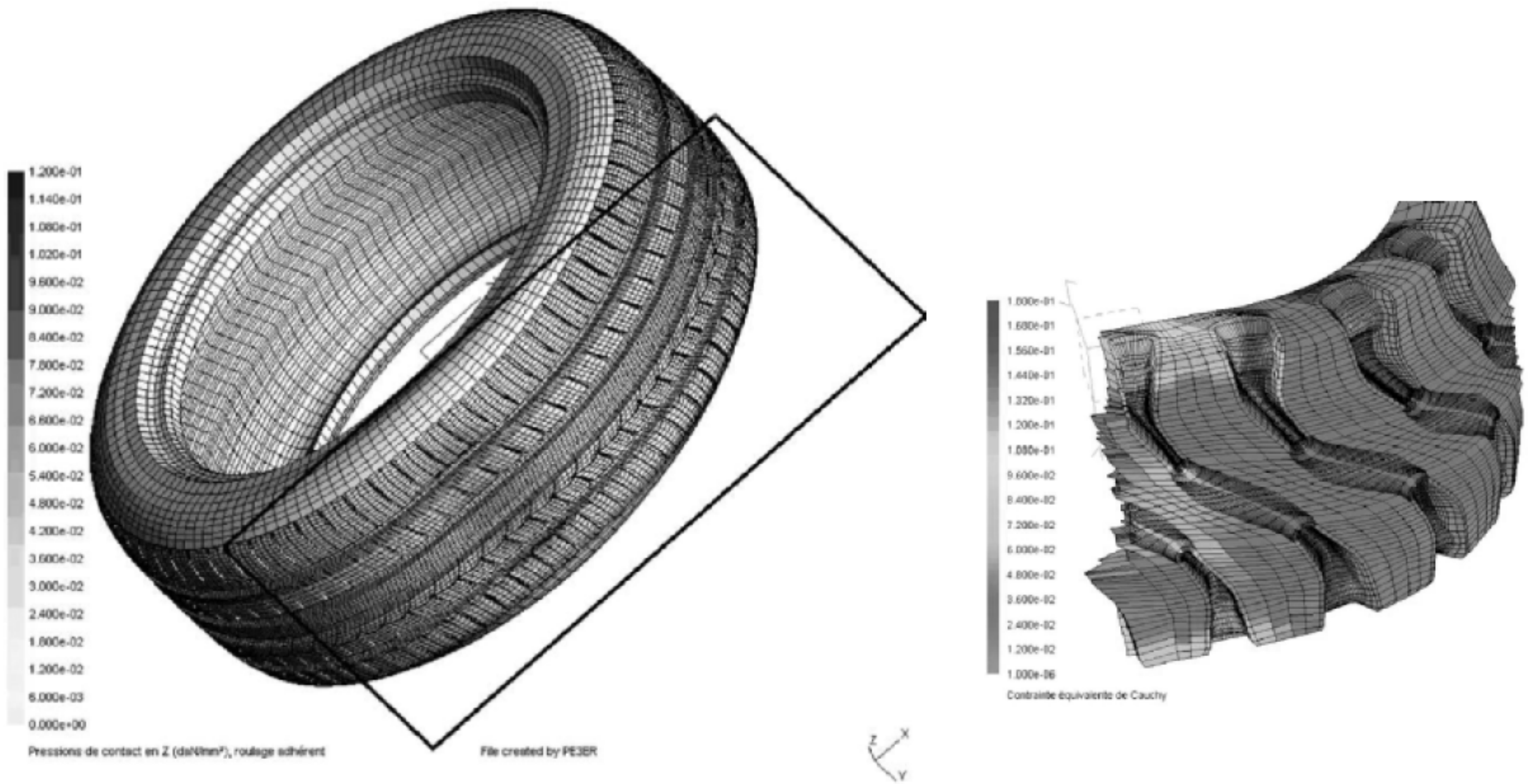
Self-contact of a car component during crash,



Crash of a car against a deformable barrier

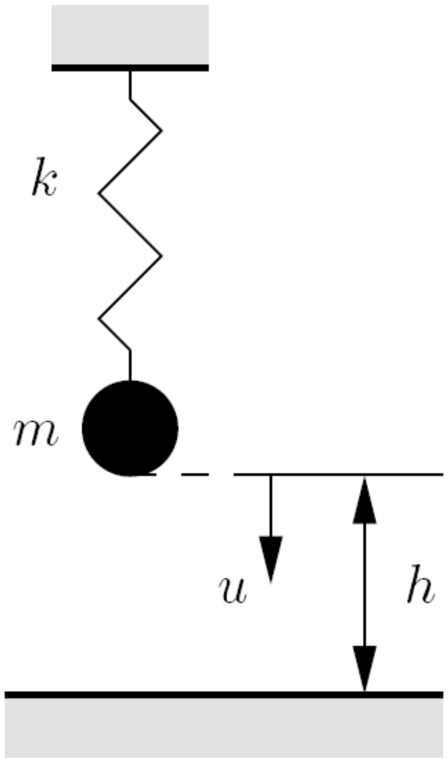
Introduction to Contact Mechanics

Contact of tyres with a road surface,

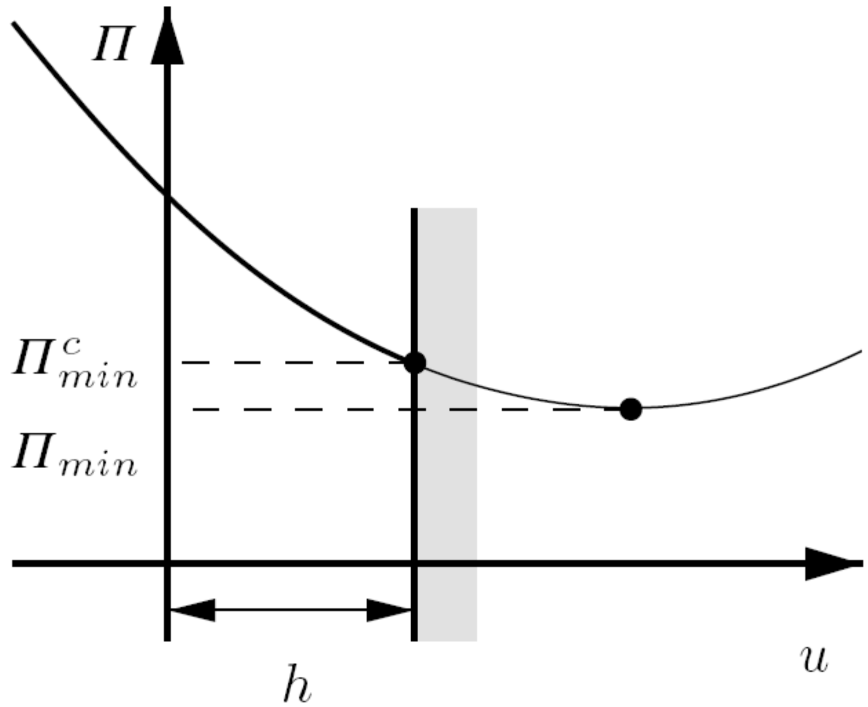


Introduction to Contact Mechanics

Contact in a Mass Spring System



Point mass supported by spring

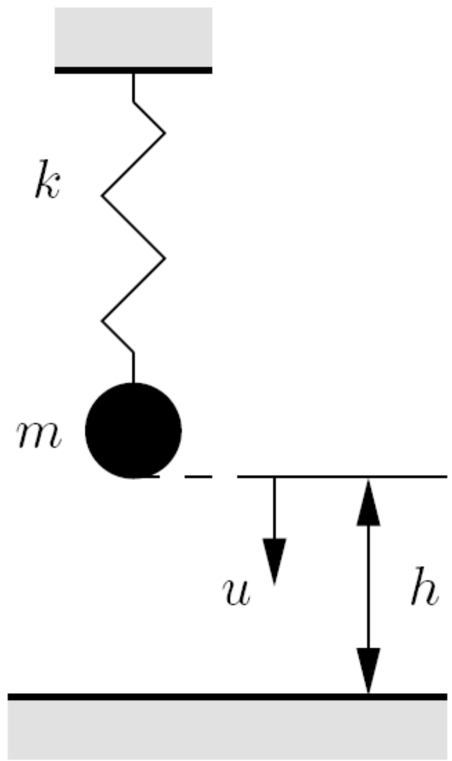


Energy of the mass spring system

Contact in a Mass Spring System

The Energy of system

$$\Pi(u) = \frac{1}{2}\, k\, u^2 - m\, g\, u$$



The variation of Energy

$$\delta\, \Pi(u) = k\, u\, \delta u - m\, g\, \delta u = 0$$

The second variation of Π

$$\delta^2\, \Pi = k$$

The minimum of u

$$u = \frac{m\, g}{k}$$

The restriction of the motion of the mass by a rigid support

$$c(u) = h - u \geq 0$$

Contact in a Mass Spring System

the point mass contacts the rigid surface, a reaction force R_N appears.

$$R_N \leq 0$$

two cases within a contact problem

$$c(u) > 0 \quad \text{and } R_N = 0$$

$$c(u) = 0 \quad \text{and } R_N < 0$$

Both cases can be combined

$$c(u) \geq 0\,, \quad R_N \leq 0 \quad \text{and } R_N\, c(u) = 0$$

Kuhn–Tucker condition

$$c(u) = h - u \geq 0$$

Contact in a Mass Spring System

Lagrange multiplier method

the Lagrange multiplier method adds to the energy o the system a term which contains the constraint

$$\Pi(u\,,\lambda) = \frac{1}{2}\,k\,u^2 - m\,g\,u + \lambda\,c(u)$$

the Lagrange multiplier λ is equivalent to the reaction force R_N .

$$k\,u\,\delta u - m\,g\,\delta u - \lambda\,\delta u = 0$$

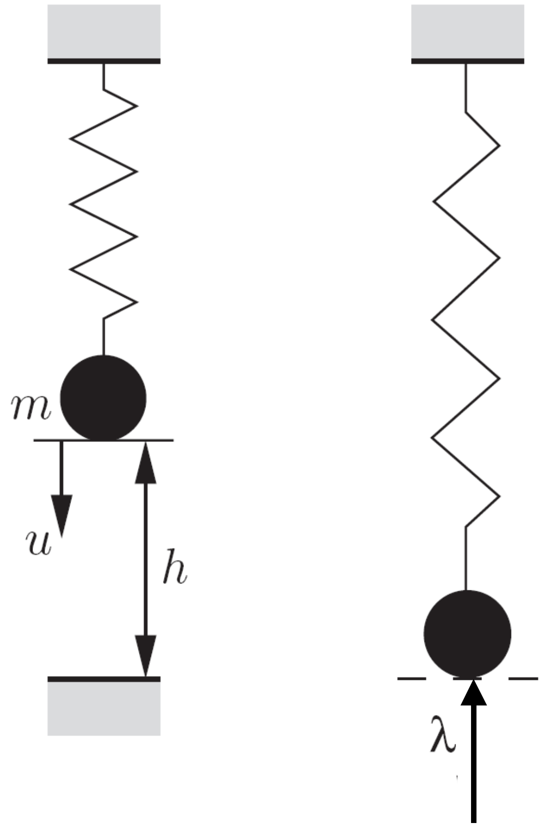
$$c(u)\,\delta\lambda = 0$$

for contact: u = h

$$\lambda = k\,h - m\,g = R_N$$

for does not contact: R_N =0

u=mg/k



Contact in a Mass Spring System

Penalty method

or an active constraint one adds a penalty term to the energy

$$\Pi(u) = \frac{1}{2}\,k\,u^2 - m\,g\,u + \frac{1}{2}\,\epsilon\,[c(u)]^2 \quad \text{with } \epsilon > 0$$

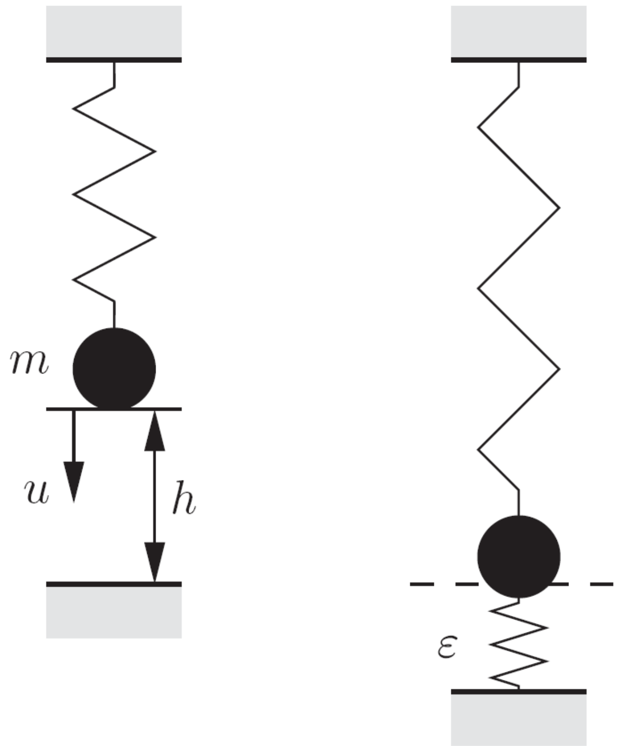
The variation of the energy

$$k\,u\,\delta u - m\,g\,\delta u - \epsilon\,c(u)\,\delta u = 0$$

the solution

$$u = (\,m\,g + \epsilon\,h)\,/\,(k + \epsilon)$$

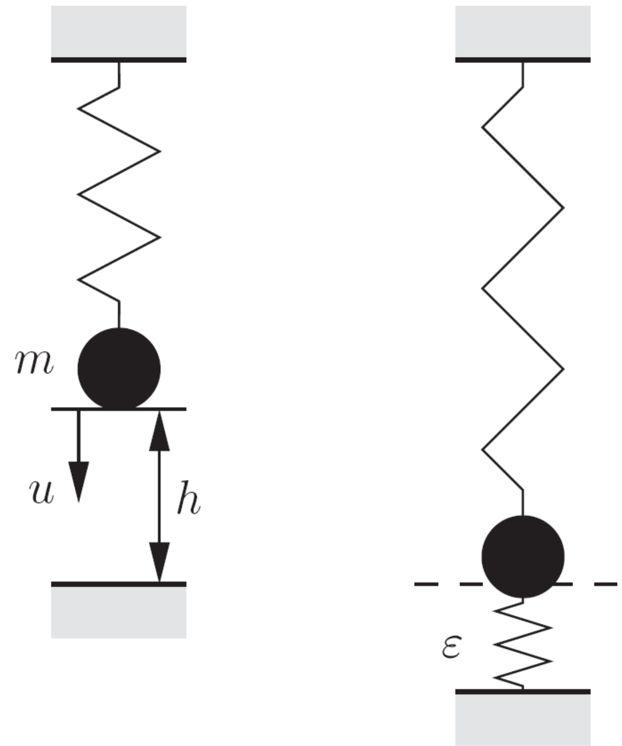
$$c(u) = h - u = \frac{k\,h - m\,g}{k + \epsilon}$$



penalty spring due to the penalty term

Contact in a Mass Spring System

Penalty method $$c(u) = h - u = \frac{k\,h - m\,g}{k + \epsilon}$$



in the contact, $$mg \geq kh$$

Penetration of the point mass into the rigid support occurs.

$$\epsilon \;\longrightarrow\; \infty \;\Longrightarrow\; c(u) \;\longrightarrow\; 0$$

$\epsilon \;\longrightarrow\; 0$ *the unconstrained solution.*

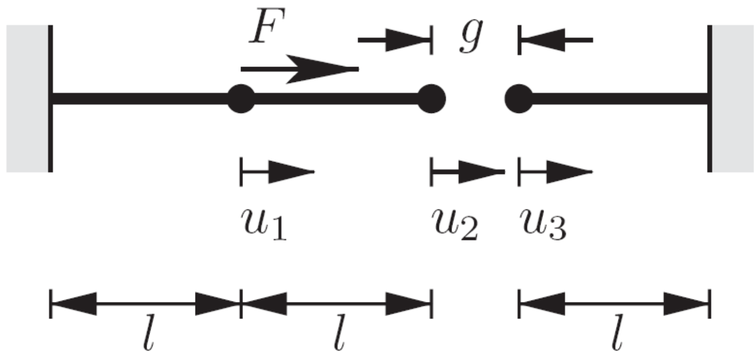
The reaction force

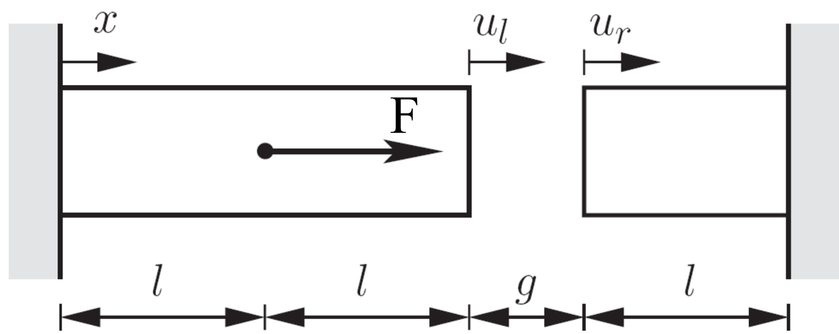
$$R_N = \lambda = \epsilon\, c(u) = \frac{\epsilon}{k+\epsilon}\,(\,k\,h - m\,g\,)$$

$\epsilon \;\longrightarrow\; \infty$ the correct solution obtained with the Lagrange multiplier method. $$\lambda = k\,h - m\,g = R_N$$

Finite Element Analysis of the Contact of Two Bars

Lagrange method





$$\Pi = \frac{1}{2} \int_{(l)} EA \, [u'(x)]^2 \, dx - \sum_i F_i \, u(x_i)$$

$$u_l - u_r \leq g$$

$$\begin{aligned}
0 \leq x \leq l \quad &: u(x) = \tfrac{x}{l} \, u_1 & u'(x) &= \tfrac{u_1}{l} \, , \\
l < x \leq 2l \quad &: u(x) = (2 - \tfrac{x}{l}) \, u_1 + (\tfrac{x}{l} - 1) \, u_2 & u'(x) &= -\tfrac{u_1}{l} + \tfrac{u_2}{l} \\
2l < x \leq 3l &: u(x) = (3 - \tfrac{x}{l}) \, u_3 & u'(x) &= -\tfrac{u_3}{l} \, .
\end{aligned}$$

Finite Element Analysis of the Contact of Two Bars

Lagrange method $$\mathit{\Pi} = \frac{1}{2} \int_{(l)} EA\,[u'(x)]^2\,dx - \sum_i F_i\,u(x_i)$$

$$\mathit{\Pi} = \frac{1}{2}\,\frac{EA}{l}\,\left[\,u_1^2 + (u_2 - u_1)^2 + u_3^2\,\right] - F\,u_1$$

$$\delta\mathit{\Pi} = \frac{EA}{l}\,[\,u_1\,\delta u_1 + (u_2 - u_1)\,(\delta u_2 - \delta u_1) + u_3\,\delta u_3\,] - F\,\delta u_1 = 0$$

$$u_2 - u_3 \leq g$$

For $u_2 - u_3 < g$

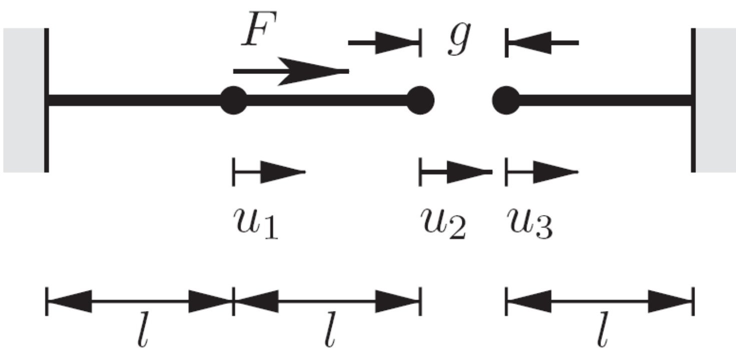
$$\langle \delta u_1\,, \delta u_2\,, \delta u_3 \rangle \left\{ \begin{array}{c} \frac{EA}{l}\,(2u_1 - u_2) - F \\ \frac{EA}{l}\,(u_2 - u_1) \\ \frac{EA}{l}\,u_3 \end{array} \right\} = 0$$

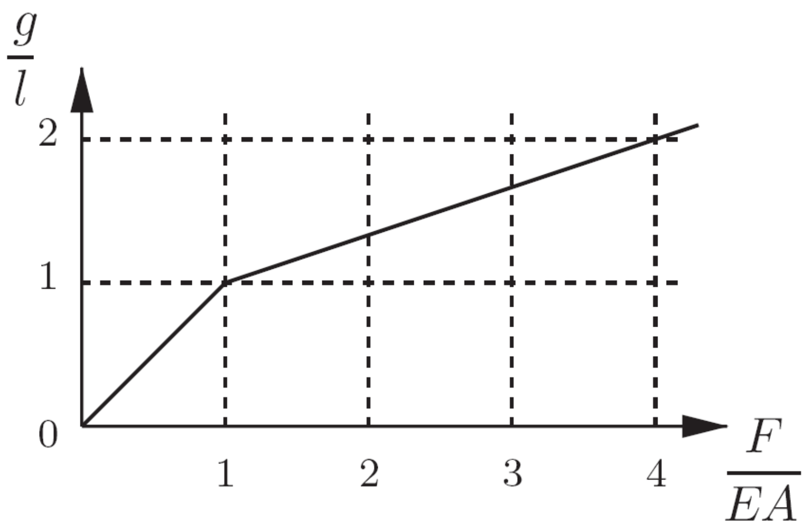
Finite Element Analysis of the Contact of Two Bars

Lagrange method

$$\frac{EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}$$

$$u_1 = u_2 = \frac{Fl}{EA}, \quad u_3 = 0$$





Finite Element Analysis of the Contact of Two Bars

Lagrange method

$$F > EA\,\tfrac{g}{l} \qquad\qquad u_2 - u_3 = g$$

$$\Pi_{LM} = \Pi + \lambda\, g = \Pi + \lambda\,(g + u_3 - u_2)$$

$$\delta\Pi_{LM} = \delta\Pi + \lambda\,(\delta u_3 - \delta u_2) + \delta\lambda\,(g + u_3 - u_2) = 0$$

$$\langle \delta u_1\,, \delta u_2\,, \delta u_3\,, \delta\lambda \rangle \left\{ \begin{array}{c} \frac{EA}{l}\,(2u_1 - u_2) - F \\ \frac{EA}{l}\,(u_2 - u_1) - \lambda \\ \frac{EA}{l}\,u_3 + \lambda \\ g + u_3 - u_2 \end{array} \right\} = 0$$

Finite Element Analysis of the Contact of Two Bars

Lagrange method

$$\begin{bmatrix} 2\,\frac{EA}{l} & -\frac{EA}{l} & 0 & 0 \\ -\frac{EA}{l} & \frac{EA}{l} & 0 & -1 \\ 0 & 0 & \frac{EA}{l} & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \\ -g \end{Bmatrix}$$

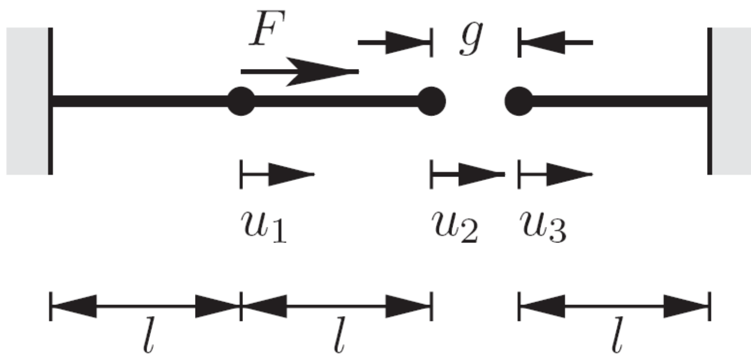
$$u_2 = \frac{1}{3}\left(2g + \frac{F\,l}{EA}\right) \qquad \lambda = \frac{1}{3}\left(EA\,\frac{g}{l} - F\right)$$

$$F = EA\left(3\,\frac{u_2}{l} - 2\,\frac{g}{l}\right)$$

Finite Element Analysis of the Contact of Two Bars

Penalty method

$$\begin{bmatrix} 2\dfrac{EA}{l} & -\dfrac{EA}{l} & 0 \\ -\dfrac{EA}{l} & \dfrac{EA}{l}+\varepsilon & -\varepsilon \\ 0 & -\varepsilon & \dfrac{EA}{l}+\varepsilon \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F \\ \varepsilon g \\ -\varepsilon g \end{Bmatrix}$$



$$u_2 = \frac{\dfrac{EA}{l}F + \left(\dfrac{2EA}{l}g + F\right)\varepsilon}{\left(\dfrac{EA}{l}\right)^2 + \dfrac{3EA}{l}\varepsilon}$$

$$\varepsilon \to \infty$$

$$u_2 = \frac{\dfrac{2EA}{l}g + F}{\dfrac{3EA}{l}} = \frac{1}{3}\left(\frac{Fl}{EA} + 2g\right)$$