To illustrate this point we shall consider two-dimensional elasticity problems using linear and parabolic serendipity quadrilateral elements with one- and four-point quadratures respectively.

Here at each integrating point *three* independent 'strain relations' are used and the total number of independent relations equals 3 × (number of integration points). The number of unknowns **a** is simply 2 × (number of nodes) less restrained degrees of freedom.

In Fig. 9.14(*a*) and (*b*) we show a single element and an assembly of two elements supported by a minimum number of specified displacements eliminating rigid body motion. The simple calculation shows that only in the assembly of the quadratic elements is elimination of singularities possible, all the other cases remaining strictly singular.

In Fig. 9.14(*c*) a well-supported block of both kinds of elements is considered and here for both element types non-singular matrices may arise although local, near singularity may still lead to unsatisfactory results (see Chapter 10).

The reader may well consider the same assembly but supported again by the minimum restraint of three degrees of freedom. The assembly of linear elements with a single integrating point *will* be singular while the quadratic ones will, in fact, usually be well behaved.

For the reason just indicated, linear single-point integrated elements are used infrequently in static solutions, though they do find wide use in 'explicit' dynamics codes – but needing certain remedial additions (e.g., hourglass control[21,22]) – while four-point quadrature is often used for quadratic serendipity elements.†

In Chapter 10 we shall return to the problem of convergence and will indicate dangers arising from local element singularities.

However, it is of interest to mention that in Chapter 12 we shall in fact *seek* matrix singularities for special purposes (e.g., incompressibility) using similar arguments.

9.12 Generation of finite element meshes by mapping. Blending functions

It would have been observed that it is an easy matter to obtain a coarse subdivision of the analysis domain with a small number of isoparametric elements. If second- or third-degree elements are used, the fit of these to quite complex boundaries is reasonable, as shown in Fig. 9.15(*a*) where four parabolic elements specify a sectorial region. This number of elements would be too small for analysis purposes *but a simple subdivision into finer elements* can be done automatically by, say, assigning new positions of nodes of the central points of the curvilinear coordinates and thus deriving a larger number of similar elements, as shown in Fig. 9.15(*b*). Indeed, automatic subdivision could be carried out further to generate a field of triangular elements. The process thus allows us, with a small amount of original *input data*, to derive a finite element mesh of any refinement desirable. In reference 23 this type of mesh generation is developed for two- and three-dimensional solids and surfaces and is reasonably

† Repeating the test for quadratic lagrangian elements indicates a singularity for 2 × 2 quadrature (see Chapter 10 for dangers).
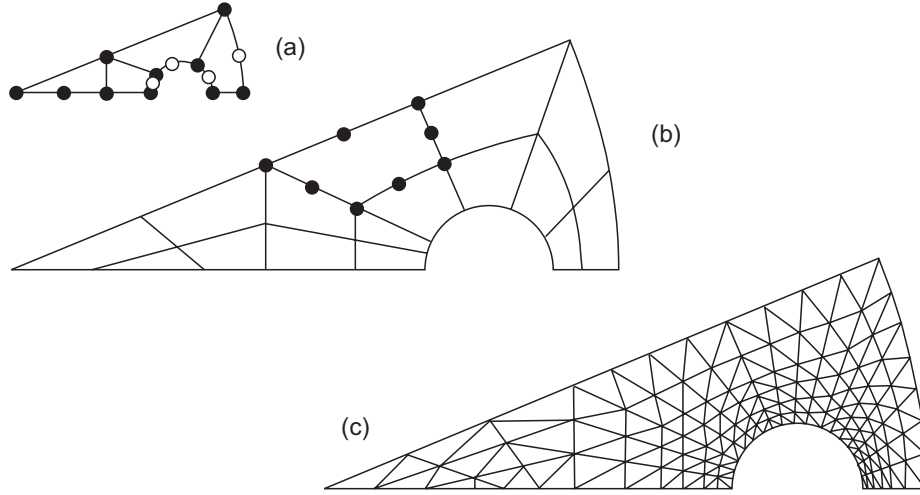
Fig. 9.15 Automatic mesh generation by parabolic isoparametric elements. (a) Specified mesh points. (b) Automatic subdivision into a small number of isoparametric elements. (c) Automatic subdivision into linear triangles.

efficient. However, elements of predetermined size and/or gradation cannot be easily generated.

The main drawback of the mapping and generation suggested is the fact that the originally circular boundaries in Fig. 9.15(*a*) are approximated by simple parabolae and a geometric error can be developed there. To overcome this difficulty another form of mapping, originally developed for the representation of complex motor-car body shapes, can be adopted for this purpose.[24] In this mapping blending functions interpolate the unknown u in such a way as to satisfy *exactly* its variations along the edges of a square ξ, η domain. If the coordinates x and y are used in a parametric expression of the type given in Eq. (9.1), then any complex shape can be mapped by a single element. In reference 24 the region of Fig. 9.15 is in fact so mapped and a mesh subdivision obtained directly without any geometric error on the boundary.

The blending processes are of considerable importance and have been used to construct some interesting element families[25] (which in fact include the standard serendipity elements as a subclass). To explain the process we shall show how a function with prescribed variations along the boundaries can be interpolated.

Consider a region $-1 \leqslant \xi, \eta \leqslant 1$, shown in Fig. 9.16, on the edges of which an arbitrary function ϕ is specified [i.e., $\phi(-1, \eta), \phi(1, \eta), \phi(\xi, -1), \phi(\xi, 1)$ are given]. The problem presented is that of interpolating a function $\phi(\xi, \eta)$ so that a smooth surface reproducing precisely the boundary values is obtained. Writing

$$N^1(\xi) = \frac{1 - \xi}{2} \qquad N^2(\xi) = \frac{1 + \xi}{2}$$

$$N^1(\eta) = \frac{1 - \eta}{2} \qquad N^2(\eta) = \frac{1 + \eta}{2} \tag{9.45}$$

for our usual one-dimensional linear interpolating functions, we note that

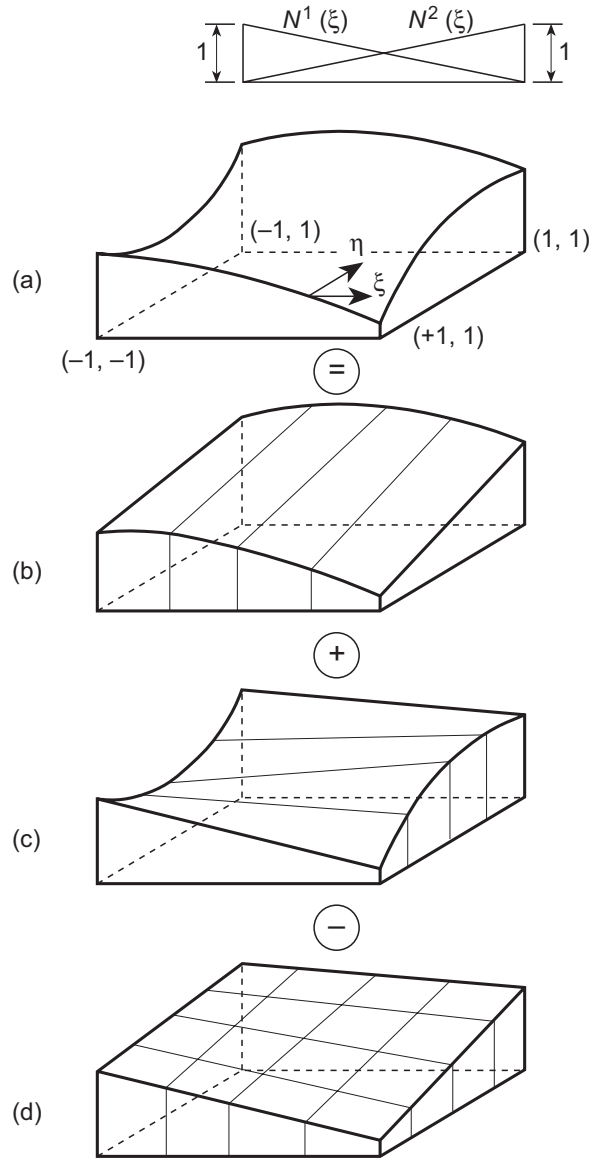$$P_\eta \phi \equiv N^1(\eta)\phi(\xi, -1) + N^2(\eta)\phi(\xi, 1) \tag{9.46}$$

Fig. 9.16 Stages of construction of a blending interpolation (a), (b), (c), and (d).

interpolates linearly between the specified functions in the η direction, as shown in Fig. 9.16(b). Similarly,

$$P_\xi \phi \equiv N^1(\xi)\phi(\eta, -1) + N^2(\xi)\phi(\eta, 1) \tag{9.47}$$

interpolates linearly in the ξ direction [Fig. 9.16(c)]. Constructing a third function which is a standard linear, bilinear interpolation of the kind we have already encountered [Fig. 9.16(d)], i.e.,

$$P_\xi P_\eta \phi = N^2(\xi)N^2(\eta)\phi(1,1) + N^2(\xi)N^1(\eta)\phi(1,-1)$$
$$+ N^1(\xi)N^2(\eta)\phi(-1,1) + N^1(\xi)N^1(\eta)\phi(-1,-1) \tag{9.48}$$

we note by inspection that

$$\phi = P_\eta\phi + P_\xi\phi - P_\xi P_\eta\phi \tag{9.49}$$

is a smooth surface interpolating exactly the boundary functions.

Extension to functions with higher order blending is almost evident, and immediately the method of mapping the quadrilateral region $-1 \leqslant \xi, \eta \leqslant 1$ to any arbitrary shape is obvious.

Though the above mesh generation method derives from mapping and indeed has been widely applied in two and three dimensions, we shall see in the chapter devoted to adaptivity (Chapter 15) that the optimal solution or specification of *mesh density* or *size* should guide the mesh generation. We shall discuss this problem in that chapter to some extent, but the interested reader is directed to references 26, 27 or books that have appeared on the subject.[28−31] The subject has now grown to such an extent that discussion in any detail is beyond the scope of this book. In the programs mentioned at the end of each volume of this book we shall refer to the GiD system which is available to readers.[32]

9.13 Infinite domains and infinite elements

9.13.1 Introduction

In many problems of engineering and physics infinite or semi-infinite domains exist. A typical example from structural mechanics may, for instance, be that of three-dimensional (or axisymmetric) excavation, illustrated in Fig. 9.17. Here the problem is one of determining the deformations in a semi-infinite half-space due to the removal of loads with the specification of zero displacements at infinity. Similar problems abound in electromagnetics and fluid mechanics but the situation illustrated is typical. The question arises as to how such problems can be dealt with by a method of approximation in which elements of decreasing size are used in the modelling process. The first intuitive answer is the one illustrated in Fig. 9.17(*a*) where the infinite boundary condition is specified at a finite boundary placed at a *large distance* from the object. This, however, begs the question of what is a 'large distance' and obviously substantial errors may arise if this boundary is not placed far enough away. On the other hand, pushing this out excessively far necessitates the introduction of a large number of elements to model regions of relatively little interest to the analyst.

To overcome such 'infinite' difficulties many methods have been proposed. In some a sequence of nesting grids is used and a recurrence relation derived.[33,34] In others a boundary-type exact solution is used and coupled to the finite element domain.[35,36] However, without doubt, the most effective and efficient treatment is the use of 'infinite elements'[37−40] pioneered originally by Bettess.[41] In this process the conventional, finite elements are coupled to elements of the type shown in Fig. 9.17(*b*) which model in a reasonable manner the material stretching to infinity.

The shape of such two-dimensional elements and their treatment is best accomplished by mapping[39−41] these onto a bi-unit square (or a finite line in one dimension or cube in three dimensions). However, it is essential that the sequence of trial